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EGALITARIANISM AND UTILITARIANISM IN QUASI-LINEAR BARGAINING

by

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Introduction

Equal sharing of the cooperative surplus is perhaps the simplest notion of micro-economic justice. In many contents, however, this principle is hardly operational: without an objective numeraire, interpersonal comparisons of welfare increments are not possible.

We assume here that all agents have constant marginal utility for money and side payments (unconstrained monetary transfers among agents) are feasible. Individual utilities are all measured in money, thus Pareto efficiency is tantamount to maximizing the joint utility. In particular requiring efficiency does not contradict any redistributive goal (no equity-efficiency dilemma).

Formally we assume that agent i's utility takes the form $u_i(a) + t_i$ where t_i is his monetary endowment and a summarizes all relevant nonmonetary parameters; the underlying microeconomic rationality has been carefully analyzed by BEWLEY [1977]: agents are endowed with a permanent income and their environment does not fluctuate too much; in our context this means that the surplus to be shared is small realtive to each individual's monetary reserves. We call it (following GREEN-LAFFONT [1979]) the quasi-linear context.

In our economy, we have two goods, one pure private good (money) with zero initial endowment and one pure public good. An allocation is a public decision a and a vector $(t_1,\dots,t_n)$ of monetary transfers (constrained by $t_1 + \dots + t_n = 0$). The set of feasible public decisions is taken to be finite, but this assumption can be easily relaxed (comment A in Section 7).

To compare the individual surplus at various public decisions one must specify the reference utility level from which these increments are computed.

If a particular public decision is viewed as the status quo (or disagreement decision) then our reference will be the corresponding utility levels. A more general method goes by picking any fixed convex combinations of the public decisions as the (generalized) status quo. For instance, to avoid any discrimination among public decisions, we can take agent i's reference utility level to be his average utility over public decisions (this method was originally proposed by DUBINS [1977]). To any such convex combination is then associated an egalitarian social choice function.

The above construction can be generalized by letting the convex average of public decisions, from which the reference level is computed, depend upon the overall joint utility: this yields the equal sharing above a convex decision social choice functions (in short E S C D). Both the egalitarian and the E S C D social choice functions will be illustrated in the particular case of a binary public decision (Section 1) and defined with full generality in Section 2. For the time being we mention the other prominent examples of E S C D social choice functions (besides egalitarian s.c.fs.) namely the utilitarian s.c.fs, that merely select the efficient public decision and perform no redistributive transfer at all (at least when the efficient public decision is unique).

In the quasi-linear context, we characterize axiomatically egalitarianism, utilitarianism and, in general, the class of E S C D social choice functions. Our two main axioms are as follows:

- No transfer paradox: no individual agent can benefit from an ex ante contingent monetary gift to one or more of his fellow agents.

- No advantageous reallocations: no coalition of agents can enter ex ante into an enforceable contract specifying certain monetary transfers within the coalition, contingent upon the final public decision, so as to improve upon the final utility of all coalized agents.

Both axioms prevent specific tactical maneuvers by which the agents try to distort the arbitration process for their own sake. An advantageous reallocation is coalitional insurance of the form: I give you \$1 if decision a occurs while you give me \$2 if b occurs. This in turn modifies our utility functions and, through the social choice function, the final allocation. Similarly a paradoxical transfer is a gift contingent upon the public decision such as: I give you \$1 if a occurs and nothing otherwise. This, again, affects the final allocation (via the social choice function) and might prove beneficial to the donor. These manipulations, however, radically differ from the familiar misrepresentation of preferences: a paradoxical transfer (namely a beneficial one) and/or an advantageous reallocation should both take place publicly: the whole point is to have the referee acknowledge a skillful change in the utility profile, so as to induce a favorable change in the redistributive transfers. In the framework of exchange economies these two kinds of manipulations are known to threaten the competitive equilibrium arbitration method for many preference profiles configuration (see GALE [1974], GUESNERIE and LAFFONT [1978], BALASKO [1978], CHICHILNISKY [1980] and POLEMARCHAKIS [1982]).

Here in the quasi-linear context is turns out (Theorem 1, Section 3) that the No Transfer Paradox and the No Advantageous Reallocation axioms together characterize the class of E S C D social choice functions.

The characterization of the egalitarian social choice functions is even simpler. If a (convex) status quo is exogenously given and the s.c.f. is required to guarantee the corresponding utility level to each player (this is the Individual Rationality axiom ; see Section 4) then No Advantangeous Reallocation alone suffices (corollary to Theorem 2, Section 4). On the other hand, the combination of No Advantageous Reallocation and the following No Disposal of Utility axiom characterizes <u>all</u> egalitarian s.c.f.s (Theorem 2, Section 4), no matter what is their status quo.

- <u>No disposal of utility</u>: an agent cannot benefit from contingently throwing away some of his own utility.

This axiom rules out blackmail threats of the form: I burn some of my money if this public decision is undertaken. Notice the analogy of profitable disposal of utility with profitable destruction of one's endowments in exchange economies where the competitive equilibrium arbitration prevails (see AUMANN and PELEG [1974]).

To characterize utilitarianism (Theorem 3, Section 5) the No Disposal of Utility axiom is replaced by the <u>Dummy</u> axiom: it says that an unconcerned agent (one who values equally all public decisions) should not get any share of a surplus to which he contributes nothing. Alternatively, utilitarianism is characterized by the combination of the No Transfer Paradox and the <u>Indifference to Merging and Splitting</u> axioms. This last axiom compares social choice functions for societies of variable size. It says that a coalition of any size t expects the same joint utility whether it goes to the referee as t different units or as a single unit (with summed utility) in therefore smaller society.

Our last result (Theorem 4, Section 6) singles out Egalitarianism and Utilitarianism from the class of E S C D social choice functions by focusing on secure utility levels, namely the maximin utility level that an agent is guaranteed of whatever are the preferences of his fellow agents. It turns out that Utilitarian and Egalitarian social choice functions respectively provide the lowest and highest secure utility levels within this class, which supports even more strongly the case for egalitarian arbitration methods.

Finally in Section 7 we comment upon several extensions of the model and its relation to the literature.

2) The Case of Binary Choices

We assume throughout this section that only two public decisions, denoted 0, 1, are at stake. In this simple case the egalitarian, utilitarian and E S C D social choice functions are easily described.

Agent i's preferences are determined by the utility increment from decision 0 to decision 1 namely $s_i = u_i(1) - u_i(0)$. Denote by $s_N = \sum_{i=1}^{n} s_i$ the total joint increment from decision 0 to decision 1. Thus $s_N > 0$ (respectively, $s_N < 0$) means that 1 (respectively 0) is the unique efficient decision; if $s_N = 0$ both decisions are efficient.

A social choice function specifies for each profile $(s_1,\ldots,s_n)$ an efficient public decision and a vector of transfers $(t_1,\ldots,t_n)$.

Suppose first that decision 0 is the status quo from which the available surplus (if any) is to be equally split. This gives the egalitarian s.c.f. ${}^{0}S$:

- if decision 0 is efficient ($s_N \leq 0$) then no redistributive transfers take place: a = 0 and $t_1 = \ldots = t_n = 0$.

- if decision 1 is the unique efficient public decision ($s_N > 0$) then $a = 1$ and the transfer t_i to agent i is chosen so as to insure that his surplus $u_i(1) + t_i - u_i(0) = s_i + t_i$ is the same as any other agent's surplus. Taking $t_1 + \dots + t_n = 0$ into account, this gives:

$$t_i = \frac{1}{n} s_N - s_i .$$

Suppose next that none of the public decisions emerges as a natural status quo (e.g., two town councils must decide upon the location of a facility to be used by both). Then a fair device is to take $\frac{1}{2}[u_i(0) + u_i(1)]$ as the reference utility level to agent i (one possible interpretation being to toss a fair coin between the two public decisions; yet the model developed in this paper is entirely nonprobabilistic). Computing the transfers so as to equalize the individual surplus we get the egalitarian s.c.f. ${}^{\frac{1}{2}}S$, namely:

- if decision 1 is an efficient decision then $a = 1$ and:

$$u_i(1) + t_i - \frac{1}{2}[u_i(1) + u_i(0)] = u_j(1) + t_j - \frac{1}{2}[u_j(1) + u_j(0)] \text{ all } i,$$

so that $t_i = \frac{1}{2} [\frac{1}{n} s_N - s_i]$.

- if decision 0 is an efficient decision then $a = 0$ and $t_i = \frac{1}{2} [s_i - \frac{1}{n}s_N]$.

Notice that if both decisions are efficient both allocations ($a = 1$, $t_i = -\frac{1}{2} s_i$ or $a = 0$, $t_i = \frac{1}{2} s_i$) give the same utility to each agent (namely $\frac{1}{2}[u_i(0) + u_i(1)]$) so it does not matter which one is chosen.

All egalitarian s.c.f.s obtain as follows: pick λ, $0 \leq \lambda \leq 1$ and take the reference utility level to be $\lambda u_i(1) + (1-\lambda)u_i(0)$. By equating individual surplus from that level, we obtain the s.c.f. ${}^{\lambda}S$:

$$\left.\begin{array}{l} \text{- if 1 is an efficient decision } (s_N \geq 0) \text{ then } a = 1 \\ \quad \text{and } t_i = (1-\lambda)[\frac{1}{n} s_N - s_i] \\ \text{- if 0 is an efficient decision } (s_N \leq 0) \text{ then} \\ \quad a = 0 \text{ and } t_i = \lambda [s_i - \frac{1}{n} s_N]. \end{array}\right\} \quad (1)$$

Figure 1 about here

On Figure 1 we take 1 to be the efficient decision. When λ goes from 0 to 1, the reference utility vector moves from $(u_1,u_2)(0)$ to $(u_1,u_2)(1)$.

We describe now the E S C D social choice functions. The idea, again, is to take for agent i's reference utility level a convex combination:

$$\lambda u_i(1) + (1-\lambda)u_i(0) \qquad 0 \leq \lambda \leq 1$$

where the parameter λ now depends arbitrarily upon the joint increment s_N and depends upon nothing else. You can give more weight (or less weight) to a public decision because it generates more joint utility, but not because it gives more utility to a particular agent. For instance take $\lambda = \phi(s_N)$ where ϕ is the cumulative distribution of some symmetrical probability measure on R.

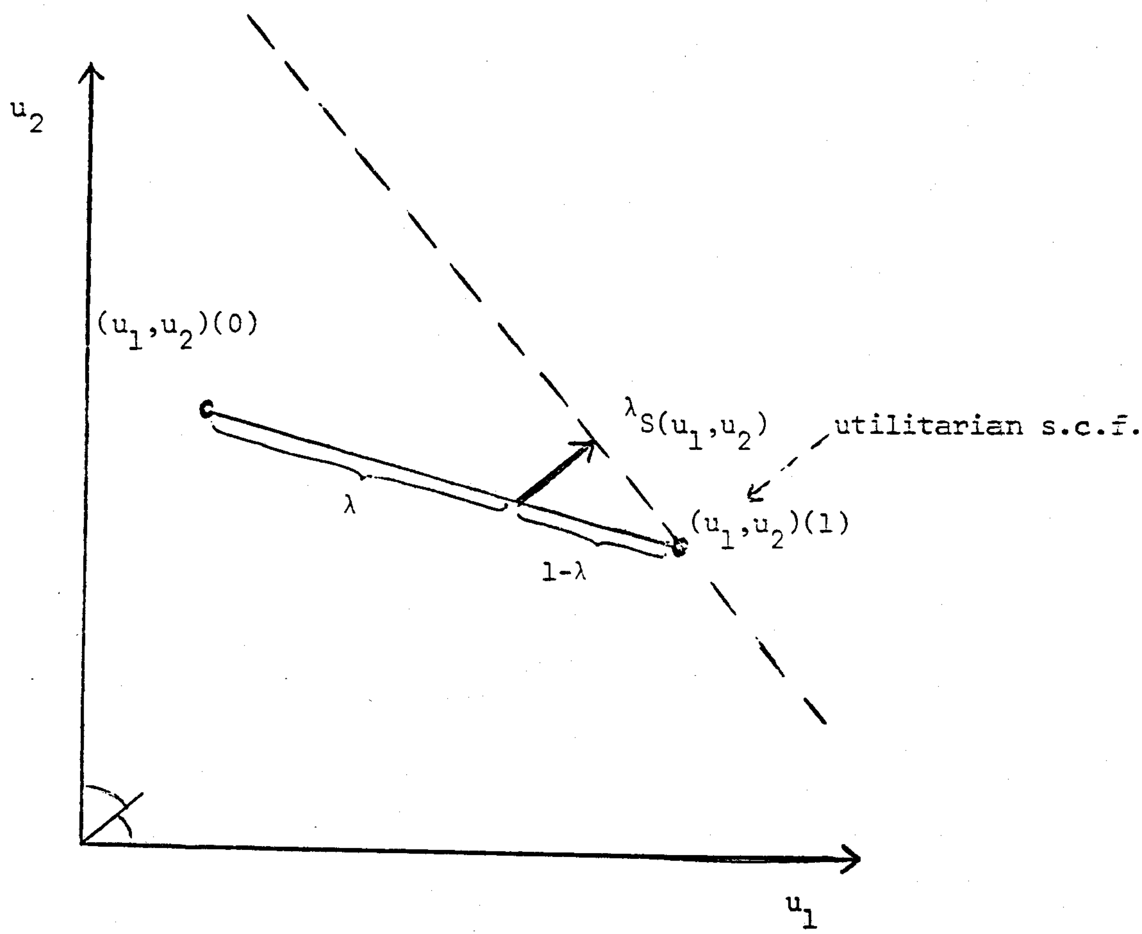

Figure 1

The Two Agents Case

Then ϕ_S is given by formulas (1) (where, now, λ depends upon s_N). for a=0,1 and b=1,0, Interpret ϕ_S as follows: if a is the unique efficient decision then the reference utility level will favor those agents who prefer a to b (they are generating the surplus, and therefore deserve a greater share of it). The larger the joint increment in utility from b to a, the closer the reference utility level will be to $u_i(a)$.

An extreme element in the family ϕ_S corresponds to a probability distribution concentrated at 0, henceforth:

$\lambda = 0$ if $s_N < 0$

$\lambda = 1$ if $s_N > 0$

$\lambda = \frac{1}{2}$ if $s_N = 0$

This is a utilitarian s.c.f. namely

- if a is the unique efficient decision, choose it and make no transfers $t_i = 0$
- if both decisions are efficient, enforce the utility level $\frac{1}{2}u_i(1) + \frac{1}{2}u_i(0)$, (e.g. by choosing $a = 0$ and $t_i = \frac{1}{2} s_i$).

Here we have utilitarianism in the vein of Poncius-Pilatus: the central planner selects the efficient decision and performs no redistributive transfers (at least when it is uniquely efficient). He "washes his hands" of the equity issue.

2. QUASI-LINEAR SOCIAL CHOICE FUNCTIONS

A quasi-linear decision problem is given by a set N of n individual agents who jointly pick a public decision a within the finite set A .

Decision a is public : no individual agent can be excluded from its consumption (although his opinion could be ignored in the choice process); it is also costless. Thus an <u>outcome</u> is a pair (a, t) where $a \in A$ and $t = (t_i)_{i \in N}$ is a vector of balanced monetary transfers : $\sum_{i \in N} t_i = 0$.

For all i in N, agent i's preferences are described by a vector u_i in R^A so that his utility for outcome (a,t) is $u_i(a) + t_i$. Denote by $\mathbf{1}$ the vector in R^A whose all components equal 1. Then two vectors u_i, v_i in R^A, such that $v_i = u_i + \alpha \mathbf{1}$ for some real number α, represent the same preferences over the outcome set and should therefore be identified. Property (3) in the following definition takes care of this by stating that the zero of an agent'sutility function plays no role.

<u>Definition 1</u>. Given a society N and a set of public decisions A, both finite, a social choice function is a mapping S associating to every preference profile $u = (u_i)_{i \in N}$ in $(R^A)^N$ a utility vector S(u) in R^N such that :

- S is efficient : $\sum_{i \in N} S_i(u) = \max_{a \in A} \{ \sum_{i \in N} u_i(a) \}$ (2)

- S is anonymous : for all permutation τ of N, all $i \in N$ and all profile u : $S_i(u^\tau) = S_{\tau(i)}(u)$ where u^τ is given by $u_i^\tau = u_{\tau(i)}$

- For all $i \in N$ and all profiles u,v such that $v_i = u_i + \alpha \mathbf{1}$ for some real α and $u_j = v_j$ for $j \neq i$, (3)

  $S_i(v) = S_i(u) + \alpha$ and $S_j(v) = S_j(u)$ all $j \neq i$.

Pick a profile u at which the maximum of $\{\sum_{i\in N} u_i\}$ is reached at a unique decision $a^\star$. By efficiency of S it follows that there is only one feasible outcome $(a^\star, t^\star)$ achieving the utility levels proposed by S. If, on the other hand, $\sum_{i\in N} u_i$ reaches its maximum at several decisions, several such outcomes exist : that the tie breaking rule should play no role is an implicit premise of Definition 1.

<u>Notation</u> : We denote $u_T = \sum_{i\in T} u_i$ and $S_T = \sum_{i\in T} S_i$ for any coalition T in N. We also denote $z^{max} = \max_{a\in A} z(a)$ for any utility z in $\mathbb{R}^A$. Thus condition (2) is rewritten as : $S_N(u) = u_N^{max}$.

<u>Egalitarian social choice function</u>

Pick an element σ in the unit simplex of $\mathbb{R}^A$

$$\sigma = (\sigma_a)_{a\in A} \quad \sum_{a\in A} \sigma_a = 1, \quad \sigma_a \geq 0 \text{ all } a$$

To each preference profile u and each agent i associate his reference utility level $u_i.\sigma = \sum_{a\in A} u_i(a)\,\sigma_a$. Above the reference vector $(u_i.\sigma)_{i\in N}$ the following surplus is left

$$u_N^{max} - \sum_{i\in N} u_i.\sigma = u_N^{max} - u_N.\sigma \geq 0$$

The egalitarian s c f associated with σ shares equally this surplus ; it is denoted ${}^\sigma S$. Henceforth :

$${}^{\sigma}S_i(u) = u_i.\sigma + \frac{1}{n}(u_N^{max} - u_N.\sigma) \text{ for all profiles } u, \text{ and agents } i \qquad (4)$$

The only egalitarian s.c.f. that does not discriminate among public decisions corresponds to the barycenter σ^* of the unit simplex ($\sigma^*_a = 1/\#A$, for all a in A). This was proposed by DUBINS [1977] as a fair decision device for quasi-linear problems.

Utilitarian social choice functions

Whenever a profile u has a unique efficient public decision a^* (i.e. for almost all profiles) an utilitarism s.c.f. selects that decision and makes no monetary transfers. If several such efficient decisions exist, we leave some freedom of choice to the utilitarian referee, namely that of achieving (by appropriate transfers) any convex combination of the efficient decisions. However the coefficients of this convex combination depend upon the joint utility u_N only : a utilitarian referee may not indirectly achieve some redistributive goal by weighing more some efficient decisions because certain agents like them more.

Formally let τ be any mapping from R^A into its unit simplex such that $z.\tau(z) = z^{max}$, all z, or equivalently :

$$\tau(z)_a > 0 \Rightarrow z(a) = z^{max} \qquad (5)$$

Then the corresponding utilitarian s.c.f. is denoted ${}^{\tau}S$ and defined :

$$^{\tau}S_i(u) = u_i \cdot \tau(u_N) \quad \text{for all profiles } u \text{ and agents } i. \tag{6}$$

In order that (6) satisfy our definition of a social choice function one must in addition satisfy the invariance axiom (3). This amounts to:

$$\tau(z + \alpha \mathbf{1}) = \tau(z) \qquad \text{all } z \text{ in } R^A, \text{ all } \alpha \text{ in } R \tag{7}$$

Equal sharing above a convex decision

This class includes the two previous ones. Let ρ be any mapping from $\mathbb{R}^A$ into its unit simplex such that

$$\rho(z + \alpha \mathbf{1}) = \rho(z) \qquad \text{all } z \text{ in } R^A, \text{ all } \alpha \text{ in } R \tag{8}$$

To ρ we associate the following E S C D social choice function :

$$S_i(u) = \frac{1}{n} u_N^{max} + (u_i - \frac{1}{n} u_N) \cdot \rho(u_N) \qquad \begin{array}{l}\text{all profile } u\\ \text{all agent } i\end{array} \tag{9}$$

Thus an average agent (i.e. $u_i = \frac{1}{n} u_N$) gets exactly his fair share of the maximal joint utility, while an non-average agent is subsidisied or taxed in proportion to his (vector) deviation $u_i - \frac{1}{n} u_N$.

A more transparent interpretation of (9) obtains by rewriting it as

$$S_i(u) = u_i \cdot \rho(u_N) + \frac{1}{n} (u_N^{max} - u_N \cdot \rho(u_N))$$

Interpret $\rho(u_N)$ as the convex decision at which the reference utility level is taken : this decision can vary as the joint utility varies but it cannot favour a particular agent or coalition of agents. Now formula (9) expresses that the surplus left over the reference utility levels, namely $u_N^{max} - u_N \cdot \rho(u_N)$ is equally shared among the agents.

Here are some examples of E S C D s.c.f.s. We impose neutrality (no discrimination among outcomes): just as $_{\sigma^*}S$ ($\sigma^*_a = 1/\#A$ for all a) is the only neutral egalitarian s.c.f., so there is one neutral utilitarian s.c.f. namely $_{\tau^*}S$ where τ^* satisfies (5) and, in addition:

$$\tau^*(z)_a = \tau^*(z)_b \quad \text{whenever} \quad z(a) = z(b) = z^{max} \ .$$

For any number λ, $0 \leq \lambda \leq 1$, the convex combination $\rho(z) = \lambda\sigma^* + (1-\lambda)\tau^*(z)$ is compromise between $_{\sigma^*}S$ and $_{\tau^*}S$. In fact λ can depend arbitrarily upon z, as long as $\lambda(z+\alpha\mathbb{1}) = \lambda(z)$ for all $z \in \mathbb{R}^A$, $\alpha \in \mathbb{R}$. For instance $\lambda = \theta(u_N^{max} - u_N \cdot \sigma^*)$ where θ is decreasing (increasing) means that the larger the average surplus the more utilitarian (the more egalitarian) is the corresponding allocation.

3. ADVANTAGEOUS TRANSFERS AND REALLOCATIONS

Definition 2. Given A , N and a social choice function S we say that S satisfies :

i) the No Transfer Paradox axiom (in short N TP), if for all profiles u, v and agent i we have

$$\{u_i < v_i \ \underline{\text{and}} \ v_j < u_j \ \text{all} \ j \neq i \ \underline{\text{and}} \ u_N = v_N\} \Rightarrow \{S_i(u) < S_i(v)\} \quad (10)$$

ii) the No Advantageous Reallocation axiom (in short N AR), if one can not find two profiles u, v and a coalition T such that :

$$\{u_j = v_j \quad \text{all } j \in N \setminus T\} \ \underline{\text{and}} \ \{u_T = v_T\} \ \underline{\text{and}} \ \{S_i(u) < S_i(v) \text{ for all } i \in T\} \quad (11)$$

The N TP axiom rules out unilateral gifts contingent upon the final choice, namely transfers of the form : if the public decision is a, agent i gives $\delta_j(a) \geqslant 0$ to agent j, $j \neq i$. Such a gift in effect changes v_i to $u_j = v_j + \delta_j$, all $j \neq i$ and v_i to $u_i - \sum_{j \neq i} \delta_j$. Property (10) says precisely that such changes can never be profitable to agent i.

The N AR axiom says that coalitional insurance against the public decision cannot be profitable to all members of that coalition. Both kinds of manipulation (unilateral gifts and coalitional reallocations) are :

i) publicly performed, since the referee must acknowledge the corresponding change of the utility profile in order that it has any effect on the final transfers;

ii) contingent upon the final choice of the public decision (by the invariance axiom (3) a constant transfer or a constant reallocation is simply added to the socially chosen outcome).

The combination of N TP and N AR proves to be quite powerful.

<u>Theorem 1</u>.

Let A, N be given. Then an E S C D social choice function (equal sharing above a convex decision, given by (9)) satisfies both the N T P and N AR axioms. Conversely suppose N contains at least three agents. Then a social choice function satisfying both N TP and N AR must be some ESCD s.c.f..

Proof

STEP 1 S satisfies NAR if and only if for all profiles u, v and all coalition T we have :

$$\{u_T = v_T \ \underline{\text{and}} \ u_j = v_j \ \text{ all } j \in N \setminus T\} \Rightarrow \{S_T(u) = S_T(v)\} \qquad (12)$$

Clearly (12) implies NAR. Conversely suppose (12) fails for some u, v and T :

$$u_T = v_T, \quad u_j = v_j \quad \text{all } j \in N \setminus T, \quad S_T(u) < S_T(v)$$

Setting $\alpha = \frac{1}{|T|} [S_T(v) - S_T(u)]$ we define

for all $i \in T$ $w_i = v_i + [S_i(u) - S_i(v) + \alpha]$.

for all $j \in N \setminus T$ $w_j = v_j = u_j$

Next we check that $w_T = v_T = u_T$ and compute by (3)

$$S_i(w) = S_i(u) + \alpha > S_i(u) \text{ for all } i \in T$$

Thus S violates NAR.

STEP 2. S satisfies NTP and NAR if and only if for all profiles u, v we have :

$$\{u_i \leq v_i \ \underline{\text{and}} \ u_N = v_N\} \Rightarrow \{S_i(u) \leq S_i(v)\} \qquad (13)$$

First of all (13) implies (10). To prove that (13) implies (12) choose two profiles u, v and a coalition T satisfying the premises of (12). Applying (13) successively with $j \in N \setminus T$ in place of 1 gives :

$$S_j(u) = S_j(v) \quad \text{all } j \in N \setminus T$$

Besides, by efficiency of S we have

$$S_N(u) = S_N(v)$$

Combining the last two equations gives $S_T(u) = S_T(v)$ as was to be proved.

We prove now the only if statement. Pick two profiles as in the premise of (13). Next construct some utility function w_j, $j \in N \setminus 1$ such that

$$w_{N \setminus 1} = v_{N \setminus 1}, \quad w_j \leq u_j \quad \text{all } j \in N \setminus 1$$

this is possible since $w_{N \setminus 1} \leq u_{N \setminus 1}$. Then we have

$$(10) \Rightarrow S_1(u) \leq S_1(v_1, w_{N \setminus 1})$$

$$(12) \Rightarrow S_{N \setminus 1}(v_1, w_{N \setminus 1}) = S_{N \setminus 1}(v) \Rightarrow S_1(v_1, w_{N \setminus 1}) = S_1(v)$$

(the last implication by the efficiency of S). This proves (13).

STEP 3. An E S C D social choice function satisfies N TP and N AR. This follows Step 2 and formula (9).

STEP 4. A s.c.f. satisfying N TP and N AR is an E S C D.

Let S be a s.c.f. satisfying N TP and N AR. By step 1 the sum $S_{N\setminus i}$ is, for all i, a function of $u_{N\setminus i}$ and u_i only. Since $S_i + S_{N\setminus i}$ depends upon u_N only, we can write S_i

$$S_i(u) = M(u_i, u_N) \quad \text{all } i \in N, \text{ all profile } u$$

where M is real valued and defined on $(R^A)^2$. Notice that the function M does not depend on i by virtue of the anonymity axiom.
Now apply property (12) to coalition $T = \{1,2\}$:

$$u_1 + u_2 = v_1 + v_2 \Rightarrow M(u_1, u_N) + M(u_2, u_N) = M(v_1, u_N) + M(v_2, u_N)$$

For all x, y, x' in $\mathbb{R}^A$, we can find, since $|N| \geq 3$, a profile u such that $u_1 = x$, $u_2 = x'$, $u_N = y$, hence :

$$M(x, y) + M(x', y) = M(0, y) + M(x + x', y) \quad \text{for all } x, x', y.$$

Setting $L(x, y) = M(x, y) - M(0, y)$ this formula just says that L is additive w.r.to x. Moreover, by efficiency of S, we have

$$\sum_{i\in N} M(u_i, u_N) = (u_N)^{max} \Rightarrow \sum_{i\in N} L(u_i, u_N) + nM(0, u_N) = (u_N)^{max}$$

Using the additivity of L w.r.to x we get :

$$M(0, y) = \frac{1}{n} y^{max} - \frac{1}{n} L(y, y)$$

and, finally,

$$S_i(u) = L(u_i, u_N) + M(0, u_N) = \frac{1}{n} u_N^{max} + L(u_i, u_N) - \frac{1}{n} L(u_N, u_N)$$

An additive function on R^A is linear with respect to rational scalars so that $\frac{1}{n} L(x, y) = L(\frac{1}{n} x, y)$. Thus we have for all profiles u

$$S_i(u) = \frac{1}{n} u_N^{max} + L(u_i - \frac{1}{n} u_N, u_N) \tag{14}$$

Apply now property (13) : we get that L is a non decreasing function of its first variable

$$x \leqslant x' \Rightarrow L(x, y) \leqslant L(x', y). \tag{15}$$

We use now the invariance axiom (3) to derive a similar invariance property of L. Take a profile u and a number α and consider profile v: $v_i = u_i + \alpha \mathbf{1}$ all i. By (3) $S_i(v) = S_i(u) + \alpha$. Applying (14) to both sides of this equality gives, after some computation:

$$L(u_i - \frac{1}{n} u_N, u_N + n\alpha \mathbf{1}) = L(u_i - \frac{1}{n} u_N, u_N) .$$

As u and α were arbitrary we conclude:

$$L(x, y+\alpha \mathbf{1}) = L(x, y) \quad \text{all } x, y \in \mathbb{R}^A, \alpha \in \mathbb{R} \tag{16}$$

Take now a profile u, a number α and consider w: $w_i = u_i + \alpha \mathbf{1}$, $w_j = u_j$ all $j \neq i$. By (3) $S_i(w) = S_i(u) + \alpha$, which combined with (14) gives:

$$L(u_i - \frac{1}{n} u_N + \frac{n-1}{n} \alpha \mathbf{1}, u_N + \alpha \mathbf{1}) = L(u_i - \frac{1}{n} u_N, u_N) + \frac{n-1}{n} \alpha .$$

Since u, α and i are arbitrary, we get

$$L(x + \alpha \mathbf{1} , y + \frac{n}{n-1}\alpha \mathbf{1}) = L(x,y) + \alpha \text{ for all } x,y \in \mathbb{R}^A , \quad \alpha \in \mathbb{R} . \qquad (17)$$

Property (16)(17) together imply:

$$L(x+ \alpha \mathbf{1},y) = L(x,y) + \alpha \text{ for all } x,y \in \mathbb{R}^A , \alpha \in \mathbb{R} .$$

The latter invariance property together with the monotonicity (15) imply that L is continuous in its first variable (in fact L is 1-lipschitzian w.r.t. the supremum norm on $\mathbb{R}^A$). Being additive and continuous the function $x \to L(x,y)$ is linear, whence L takes the form

$$L(x, y) = x . \rho (y) \text{ For all } x, y .$$

for some mapping ρ from R^A into itself. From the monotonicity and invariance of L again, we get that $\rho (y)$ is in the unit simplex of R^A and the proof of formula (9) is complete. The invariance property (8) follows from that of L .

QED

Suppose the mischievous agents in coalition T implement the following mixture of public/private monetary reallocations : first a public contract enforces the change from u_i to v_i , $i \in T$, next some private reallocations among T redistribute the joint utility $S_T(v)$. This can prove profitable to every member of T if and only if $S_T(u) < S_T(v)$.

Now step 1 implies that this two step reallocation cannot occur if the one step reallocation described by (11) is impossible.

The next two examples prove that neither of the two axioms N TP and N AR implies the other.

First take a mapping ρ from R^A into itself and define :

$$S_i(u) = \frac{1}{n} u_N^{max} + (u_i - \frac{1}{n} u_N) . \rho(u_N) \quad \text{For all profiles } u .$$

This in turn is a social choice function as soon as ρ satisfies

$$\mathbb{1} . \rho(x) = 1 \quad \text{for all } x \text{ in } R^A$$

In that case it satisfies N AR (use step 1). Next N TP holds if and only if $\rho(x)$ is a non-negative vector of R^A. Hence many examples of s.c.f.s satisfying N AR but not N TP. For instance pick τ satisfying (5) and (7) and let $\sigma^\star$ the barycenter of the unit simplex. Then define

$$\rho(u_N) = 2\tau(u_N) - \sigma^\star \Rightarrow S_i(u) = 2 u_i . \tau(u_N) - u_i . \sigma^\star - \frac{1}{n}(u_N^{max} - u_N . \sigma^\star)$$

To construct a s.c.f. satisfying N TP but not N AR is not difficult either. To each profile associate the (subadditive) game in characteristic function form

$$V(T) = u_T^{max}$$

Then take $S(u)$ to be the Shapley value of that game.

For instance if $N = \{1,2,3\}$, $S_1(u)$ is

$$S_1(u) = \frac{1}{3} u_N^{max} + \frac{1}{6} (u_{13}^{max} + u_{12}^{max} - 2u_{23}^{max}) + \frac{1}{6}(u_2^{max} + u_3^{max} = 2u_1^{max}) \qquad (18)$$

Clearly (18) defines a social choice function. To check NTP pick two profiles u, v satisfying the premises of (10) with i=1. Then for all T not containing 1 $v_T \leq u_T$, whereas for all T containing 1 : $u_T = u_N - u_{N\setminus T} \leq v_N - v_{N\setminus T} = v_T$. The right hand inequality in (10) follows. But NAR is violated since S_1 is not a function of u_1 and u_{23} only.

4. CHARACTERIZING EGALITARIAN SOCIAL CHOICE FUNCTIONS

Definition 3. Given A , N we say that the social choice function S satisfies the No Disposal of Utility axiom (in short NDU) if S_i is non decreasing w.r.t. the variable u_i: for all profiles u, v and agent i,

$$\{u_i \leq v_i \text{ and } u_j = v_j \text{ all } j \neq i\} \Rightarrow \{S_i(u) \leq S_i(v)\} \qquad (19)$$

The NDU axiom rules out profitable self punishing policies of the form : if decision a is taken I cut my left arm, if b is taken I cut my right arm . It holds true for an egalitarian s.c.f. since (4) can be rewritten as :

$$\sigma_{S_i}(u) = \frac{n-1}{n} u_i \cdot \sigma + \frac{1}{n} u_N^{max} - \frac{1}{n} u_{N\setminus i} \sigma$$

On the other hand it fails for all utilitarian s.c.f. as the following examples shows : $A = \{a, b\}$ $N = \{1, 2\}$

$$v_1(a) = 2 \quad v_1(b) = 1 \quad ; \quad u_2(a) = 0 \quad u_2(b) = 2$$

From v_1 to u_1 : $u_1(a) = 2$, $u_1(b) = -1$, the efficient decision switches from b to a and S_1 increases from 1 to 2.

Actually among E S C D social choice functions only the egalitarian s.c.f.s satisfy the N D U axiom.

Theorem 2.

Let A, N be given with $|N| \geqslant 3$ and let S be a social choice function.

i) If S satisfies the N AR and N DU axioms, then it is an egalitarian s.c.f. : for some σ in the unit simplex of R^A , $S = {}^{\sigma}S$ (see formula (4))

ii) Conversely if S is egalitarian, then it satisfies N TP, N AR and N DU.

Proof

Only statement i) needs a proof.

From Step 4 in the proof of Theorem 1 we know that a s.c.f. S satisfying N A R takes the form (14) where L is additive w.r.t. its first variable. We write now N D U : for any profile u and nonnegative $\delta_1 \in R_+^A$ we have :

$$S_1(u) \leqslant S_1(u_1 + \delta_1, u_{-1})$$

In view of (14) and denoting $\frac{1}{n} u_N = z$, this reduces to

$$z^{max} + L(u_1 - z, nz) \leqslant (z + \delta_1)^{max} + L(u_1 - z + \frac{n-1}{n} \delta_1, nz + \delta_1) \qquad (20)$$

which, by additivity of L, amounts to

$$L(u_1, nz) - L(u_1, nz + \delta_1) \leqslant B(z, \delta_1)$$

where B does not depend on u_1. This holds for any $u_1, z \in R^A$, $\delta_1 \in R_+^A$ hence the left hand term in this inequality must be identically zero : an additive function of u_1 is uniformly bounded over R^A only if it is zero everywhere. Thus $L(x, y) = L(x, y')$ as soon as $y \leqslant y'$, implying that $L(x,y)=L(x)$ does not depend upon y. Going back to inequality (20), we get

$$z^{max} + L(u_1 - z) \leqslant (z + \delta_1)^{max} + L(u_1 - z + \frac{n-1}{n} \delta_1)$$

hence by additivity of L

$$-\delta_1^{min} = \sup_{z \in R^A} z^{max} - (z + \delta_1)^{max} \leqslant L(\frac{n-1}{n} \delta_1) \qquad (21)$$

so that L is bounded from below over any bounded set of R_+^A. Combined with the additivity of L, this implies its continuity (see e.g. ACZEL [1966] Chap. 5), hence $L(x) = x \cdot \sigma$ for some σ in R^A. Inequality (21) now implies that σ is non negative and the invariance axiom that $\mathbb{1} \cdot \sigma = 1$. This completes the proof of Theorem 2.

Suppose that agent i's final utility level S_i must be bounded from below by his utility at some (convex) public decision σ, to be interpreted as an exogeneously given disagreement outcome. Then the corresponding egalitarian social choice function emerges among E S C D ones.

Corollary to Theorem 2

Given A, N and a element σ in the unit simplex of R^A, let S be social choice function satisfying the N A R and the following I R (σ) axiom :

I R (σ) : Individual Rationality above σ :

$$u_i \cdot \sigma \leqslant S_i(u) \qquad \text{all agent } i \text{, all profile } u$$

Then S equals ${}^{\sigma}S$, the egalitarian s.c.f. above σ.

Proof

From Step 4 in the proof of Theorem 1, a s.c.f. S satisfying N A R must take the form (14) for some L additive in its first variable. From individual rationality it follows easily that $L(x, z)$ is bounded from below for any fixed z :

$$\inf_{x \in R^A} L(x, z) > -\infty \quad \text{all } z \text{ in } R^A$$

Henceforth, by the same argument as in the proof of Theorem 2, L must be linear w.r.t. x namely

$$L(x, z) = x . \rho(z)$$

Going back to (14), we express the I R (σ) axiom as :

$$x . \sigma \leq \frac{1}{n} z^{max} + (x - \frac{1}{n} z) . \rho(z) \quad \text{all } x , z \in R^A$$

This implies $\rho(z) = \sigma$ all z.

Another property of egalitariam s.c.f.s that is not shared by utilitarian s.c.f.s is <u>continuity</u> (S is continuous as a mapping from $(R^A)^N$ into R^N). That a utilitarian s.c.f. is not continuous is easy to check : in the example before Theorem 2 suppose $v_1(b)$ varies continuously from + 1 to - 1 : then at $v_1(b) = 0$ both S_1 and S_2 jump. In fact an E S C D s.c.f. is continuous if and only if the mapping c is continuous. Therefore egalitarian s.c.f.s are not characterized as continuous E S C D s.c.f.s.

5. CHARACTERIZING UTILITARIAN SOCIAL CHOICE FUNCTIONS

<u>Definition 4</u>. Given A, N we say that the social choice function S satisfies the <u>Dummy</u> axiom if

$$u_1 = 0 \Rightarrow S_1(u) = 0 \quad \text{all profile } u$$

The Dummy axiom says that unconcerned agents, those who do not contribute to the cooperative surplus (from any status quo) should get no share of a cake they did not cook.

By definition (6) utilitarian s.c.f.s do satisfy the Dummy axiom. Also, egalitarian s.c.f.s violate it, since these arbitrations methods give an equal share of the surplus to all agents, regardless of the magnitude of individual contributions.

Theorem 3

Let A, N be given with $|N| \geq 3$. The social choice function S is utilitarian (there exists τ satisfying (5) and (7) such that (6) holds) if and only if it satisfies N TP, N AR and Dummy.

Proof

If S satisfies N TP and N AR and $|N| \geq 3$, then by Theorem 1 it is an E S C D s.c.f.. In view of formula (9), the Dummy axiom amounts to :

$$z \cdot \rho(z) = z^{\max} \quad \text{all } z \text{ in } R^A$$

which is an equivalent formulation of (5) (given that $\rho(z)$ is in the unit simplex of R^A).

Of the three axioms characterizing utilitarian s.c.f.s two of them, namely N AR and Dummy can be given an equivalent, more compact formulation.

Definition 5. Given A, and, for all $n \geq 1$, social choice funtions S^n for societies of size n, we say that the family $\{S^n, n=1,2$ is indifferent to merging and splitting if for all $n \geq 1$, all profile $u = (u_1, \dots, u_n)$ and all $t \leq n$ we have :

$$S^n_T(u) = S^{n-t+1}_1(u_T, u_{t+1}, \dots, u_n) \quad \text{where } T = \{1, \dots, t\}$$

Inequality $S^n_T(u) < S^{n-t+1}_1(u_T, u_{t+1}, \dots, u_n)$ means that the agents of T would gain by merging into a single agent (syndicate), later dividing amongst themselves the joint utility allocated to the syndicate in the reduced society. The reverse inequality means an incentive to split.

The Dummy axiom is now modified to account for a family of social choice functions of all sizes. We say that the family $\{S^n\}_{n \geq 1}$ satisfies the Dummy* axiom if we have :

$\{u_1 = 0\} \Rightarrow \{S_1^n(u) = 0$ and $S_i^n(u) = S_i^{n-1}(u_{-1})$ all $i \geq 2\}$ for all $n \geq 2$ and all profile u.

It is now a simple exercise, left to the reader, to check the following :

Lemma

The family $\{S^n\}_{n\geq 1}$ is indifferent to merging and splitting if and only if it satisfies the Dummy* axiom and each S^n satisfies the Non Advantageous Reallocation axiom.

Corollary

Given A and a family $\{S^n\}_{n\geq 1}$ of social choice functions for all finite societies, the two following statements are equivalent :

i) $\{S^n\}_{n\geq 1}$ is indifferent to merging and splitting and satisfies the No Transfer Paradox axiom for all $n \geq 3$.

ii) S^n is an utilitarian social choice function for each $n \geq 1$, and its tie breaking rule τ (satisfying (5) (6)) is independent of n .

Notice that in contrast with Theorem 3 we have a characterization of utilitarian s.c.f.s even for a 2 person society. The indifference to merging and splitting axiom determines the s.c.f. S^2 from S^3 by :

$$S_1^2(u_1, u_2) = S_{1\,2}^3(u_1, 0, u_2).$$

6. SECURE UTILITY LEVELS OF EGALITARIAN AND UTILITARIAN S.C.F.S

Definition 6. Given A, N and a social choice function $\dot{S}$, its secure utility level is the real valued function h defined on R^A :

$$h(u_1) = \inf_{u_{-1} \in (R^A)^{N \setminus 1}} S_1(u) \qquad \text{for } u_1 \text{ in } R^A$$

It represents the minimal utility level that an agent (any agent, in view of the anonymity axiom) is guaranteed of when endowed with a certain utility function u_1. For a maximinimizing agent, the secure utility level is the primary criterion for comparing social choice functions. Within the class of E S C D s.c.f.s, the egalitarian s.c.f.s turn out to be best in that sense and utilitarian to be worst.

Theorem 4.

Let A and N be given, with $|N| \geq 2$.

i) The secure utility level of the utilitarian s.c.f. $_\tau S$ (given by (6)) is worth :

$$_\tau h(u_1) = u_1^{\min} = \min_{a \in A} u_1(a) \quad \text{for all } u_1$$

ii) The secure utility level of the egalitarian s.c.f. $^\sigma S$ (given by (4)) is worth :

$$^\sigma h(u_1) = u_1 . \sigma \qquad \text{for all } u_1$$

iii) If S is any E S C D social choice function given by (9) with associated secure utility level h, there exists an element σ of the unit simplex of R^A such that :

$$u_1^{min} \leq h(u_1) \leq u_1 . \sigma \qquad \text{all } u_1 \text{ in } R^A \tag{22}$$

Moreover if the right hand side inequality in (22) is an equality for all u_1, then S is the egalitarian s.c.f. ${}^{\sigma}S$.

Theorem 4 states that any E S C D s.c.f. has its secure utility level weakly dominated by that of an egalitarian s.c.f., and this domination must be strict somewhere if our s.c.f. is not egalitarian. In view of Theorem 1 this yields one more characterization of egalitarian s.c.f.s :

A social choice function i) satisfies the N T P and N A R axioms and ii) has an undominated secure utility level among those satisfying i) if and only if it is egalitarian.

Proof of Theorem 4

Statement i) and ii) are obvious. Let us prove iii). Take an E S C D s.c.f. S with associated mapping ρ (see (9)) and a profile u. We note first that

$$u_i^{min} - \frac{1}{n} u_N^{max} = u_i^{min} + (-\frac{1}{n} u_N)^{min} \leq (u_i - \frac{1}{n} u_N) . \rho(u_N)$$

which in turn implies $u_i^{min} \leq S(u)$, giving the lefthand inequality in (22). To prove the right hand side, note that the secure utility level h can be written as :

$$h(u_1) = \inf_{z \in R^A} \{z^{max} + (u_1 - z) . \rho(n\,z)\} \quad (23)$$

and is therefore a concave function of u_1 (as the infimum of affine functions). Moreover

$$h(0) = \inf_{z \in R^A} \{z^{max} - z\,\rho(n\,z)\} = 0 \quad \text{(take } z = 0\text{)}$$

There exists an hyperplane supporting at 0 the graph of **h**, that is to say there exists some σ in R^A such that

$$h(u_1) \leq u_1 . \sigma \qquad \text{all } u_1$$

Now $h\,(u_1 + \alpha \mathbf{1}) = h(u_1) + \alpha$ for all u, in R^A and α in R (by (23) since $\rho(z)$ is in the unit simplex of R^A), and hence

$$h(u_1) + \alpha \leq u_1 . \sigma + \alpha\, \mathbf{1} . \sigma \qquad \text{all } u_1, \alpha$$

This implies $\mathbf{1} . \sigma = 1$. Next h is a non decreasing function of u_1 (by (23) again), and hence σ must be non negative. This concludes the proof of (22).

We prove the last statement of iii). Suppose $h(u_1) = u_1 . \sigma$ for all u_1 : does it follows that $S = {}^{\sigma}S$? Using (23) again this gives

$$u_1 . \sigma \leq z^{max} + (u_1 - z)\; \rho(z) \quad \text{for all } u_1, z$$

$$u_1 . (\sigma - \rho(z)) \leq z^{max} - z\; \rho(z) \text{ forall } u_1, z$$

The righthand side in this last inequality being independent of u_1 we get $\sigma - \rho(z) = 0$, which was to be proved.

7. Concluding Comments

A) All our results extend straightforwardly to the case where A is --say--a compact topological space and utilities vary in C(A), the space of continuous functions over A. In that case a convex decision (as σ in (4) or $\rho(u_N)$ in (9)) is simply a nonnegative linear form over C(A), continuous w.r.t. the uniform convergence topology, which is worth 1 at the constant function 1 (Radon probability measure). If A is not endowed with any topological structure and utilities vary in the space B(A) of uniformly bounded functions over A, then our characterizations of egalitarian s.c.f.s still hold (but utilitarian s.c.f.s are no longer defined), with convex decisions covering the unit simplex of the dual space $B^*(A)$.

B) The obvious way to enlarge the scope of the quasi-linear model would be to take into account cost sharing arguments. Here we simply deal with costless public decisions: this is formally equivalent to assume equal sharing of the cost across decisions. A first step in the suggested direction is taken in MOULIN [1983a] when costs are indivisible. The next step will be to assume some cooperative technology (public goods allow exclusion so that the cost of a decision depends upon the coalition which produces it) and enter the world of games in characteristic form. This is widely open to future research.

C) In GREEN [1983] a model of quasi-linear bargaining is developed which bears close similarities to ours. However, GREEN studies social choice functions that depend only upon the set of utility vectors for various public decisions (in addition this set is taken to be convex): two public decisions with the same utility vector count for no more than one (in contrast with, say, the neutral egalitarian s.c.f. above σ^*).

D) Most of the literature on quasi-linear bargaining problems deals with manipulations by misrepresentation of preferences, after the discovery of a rich family of strategy proof, truth revealing decision mechanisms by CLARKE [1971] and GROVES [1973]. However appealing these mechanisms are, they fail to reach an efficient outcome (GREEN and LAFFONT [1979]) so that, from a first best point of view, other game forms must be proposed where other equilibrium concepts can bring about full efficiency of the outcome: maximin behavior is one possible such concept (DUBINS [1977], THOMSON [1979]) sophisticated Nash equilibrium is another one (MOULIN [1981, 1982a]). Egalitari social choice functions are already known to suit nicely these two behavioral scenarios: the truth is their unique maximinimizing message whenever the fixed convex average gives positive weight to every public decision (DUBINS [197 moreover a simple auction-like game form allows for their non-cooperative implementation: see MOULIN [1982b].

As utilitarian social choice functions fail to satisfy the No Disposal of Utility axiom, they have little normative appeal. However, axiomatic results similar to those presented here can be developed to jointly characterize utilitarian s.c.f. and the (non budget-balanced) s.c.f. derived from the pivotal mechanism: see MOULIN [1983b].

REFERENCES

ACZEL, J. [1966] : Functional Equations and their Applications, Academic Press.

AUMANN, R. and B. PELEG [1974] : A Note on Gale's Example, Journal of Mathematical Economics, 1, 209-211.

BALASKO, Y. [1978] : The Transfer Problem and the Theory of Regular Economies, International Economic Review, 19, 3, 687-694.

BEWLEY, T. [1977] : The Permanent Income Hypothesis : a Theoretical Formulation, Journal of Economic Theory, 12, 2, 252-292.

CHICHILNISKY, G. [1980] : Basic Goods, the Effects of Aid and the International Economic Order, Journal of Developments Economics, 7, 505-519.

CLARKE, E.H. [1971] : Multipart Pricing of Public Goods, Public Choice, 11, 17-33.

DUBINS, L.F. [1977] : Group Decisions Devices, American Mathemat. Monthly, May, 350-356.

GALE, D. [1974] : Exchange Equilibrium and Coalitions : An Example, Journal of Mathematical Economics, 1, (1), 63-66.

GREEN, J. [1983] : A Theory of Bargaining with Monetary Transfers, D.P. n° 966, Harvard University.

GREEN, J. and J.J. LAFFONT, [1979] : Incentive in Public Decision Making, Studies in Public Economics, Vol. 1, North Holland.

GROVES, T. [1973] : Incentives in Teams, Econometrica, 41, 617-663.

GUESNERIE, R. and J.J. LAFFONT, [1978] : Advantageous Reallocations of Initial Resources, Econometrica, 46, 4, 835-842.

MOULIN, H. [1981] : Implementing Just and Efficient Decision Making, Journal of Public Economics, 16, 193-213.

MOULIN, H. [1982-a] : Non-Cooperative Implementation : A Survey of Recent Results, Mathematical Social Sciences, 3, 243-258.

MOULIN, H. [1982-b] : The Conditional Auction Mechanism for Sharing a Surplus, Forthcoming in the Review of Economic Studies.

MOULIN, H. [1983-a] : Egalitarian Surplus Sharing in Quasi-linear Problems, Mimeo, The Hebrew University of Jerusalem.

MOULIN, H. [1983-b] : Utilitarism and the pivotal Mechanism : an axiomatic comparison, Mimeo, Laboratoire d'Econométrie de l'Ecole Polytechnique, Paris.

POLEMARCHAKIS, H. [1982] : On the Transfer Paradox, Mimeo, Columbia University.

THOMSON, W. [1979] : Maximin Strategies and Elicitation of Preferences, in Aggregation and Relevation of Preferences, J.J. LAFFONT ed. Studies in Public Economics, North Holland.