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OPTIMAL GROWTH AND PARETO-OPTIMALITY

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RESUME

L'objectif de ce papier est de montrer que dans une économie intertemporelle où les agents ont des utilités récursives tout Pareto-optimum est solution d'un problème de Mc Kenzie généralisé. Un espace d'état "abstrait" est introduit: celui des couples de stock de capital et des utilités que peuvent obtenir n-1 agents à partir de ce stock de capital. Des conditions "technologiques généralisées" sont définies sur cet espace ainsi que le critère récursif. A parti des équations de Bellman et d'Euler on généralise certains résultats dynamiques connus dans le cas séparable avec un seul agent.

Mots Clés : Modèle de Mc Kenzie, préférences récursives, optimum de Pareto, dynamique, fonctions valeurs, équation de Bellman, équation d'Euler.

OPTIMAL GROWTH AND PARETO-OPTIMALITY

ABSTRACT

The purpose of this paper is to show that in a stationary intertemporal economy where agents have recursive utilities every Pareto-optimum is solution of a generalised Mc Kenzie problem. An "abstract" state space is introduced as the space of couples of capital stock and utilities that can be reached by n-1 agents from that capital stock. "Generalised technological" conditions are then defined on that abstract space as well a recursive criterion on sequences of its elements. The criterion generalises the additively separable one. As Bellman's and Euler's equations still hold, many dynamical results known for the additively separable one agent case can be generalised.

Key words: Mc Kenzie model, recursive preferences, Pareto-optimum, dynamics, value function, Bellman's equation, Euler's equation.

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Introduction.

As it is well known, Mc Kenzie's model has played a key role in the study of the dynamical properties of optimal paths in the neoclassical theory of growth. It has been used to obtain turnpike results (Mc Kenzie [1985], Scheinkman [1976]), cycle results (Benhabib and Nishimura [1985]). More recently Boldrin and Montrucchio [1986] , Deneckere and Pelikan [1986] have used it to show that the dynamics of optimal growth paths could be arbitrarely complex.

It is also well known that in an intertemporal stationary economy where agents have additively separable utilities and same discount factor, every Pareto-optimum is solution of an optimal growth problem where the criterion is a weighted sum of the utilities of the different agents. These weights turn out to be a characteristic of the Pareto-optimum.

The purpose of this paper is to show a similar result in the case where agents have stationary utilities instead of additively separable utilities. These utilities are usually called recursive utilities in the literature (Koopmans et Ali [1964], Lucas and Stokey [1984], Benhabib et Ali [1985]...). We first show that one can easily generalise Mc Kenzie's model to the case of a recursive criterion. As Bellman's and Euler's equations still hold, many dynamical results known for the additively separable case can be generalised. We then show that in a stationary intertemporal economy where agents have recursive utilities every Pareto-optimum is solution of a generalised Mc Kenzie problem. An "abstract" state space is introduced as the space of couples of capital stock and utilities that can be reached by n-1 agent from that capital stock. "Generalised technological" conditions are then defined on that abstract space as well a recursive criterion on sequences of its elements.

The paper is organised as follows:

In part one we recall the axiomatic of recursive utilities. This part generalises Koopmans et Ali [1964] and Lucas and Stokey [1984].

In part two we introduce the generalised version of Mc Kenzie's model and extend a few dynamical results on optimal paths (e.g. Mangasarian's result [1966]).

In part three we show that every Pareto-optimum is solution of a generalised Mc Kenzie model. We then use the dynamical results of part two to give properties of examples studied in the literature (Benhabib et Ali[1985], Lucas and Stokey [1984]).

I - RECURSIVE REPRESENTATION

I.1 - Following Beals and Koopmans [1969], Lucas and Stokey [1984], we introduce an <u>aggregator</u> function defined as follows.

<u>Definition I</u>

Let q be an integer. Let C be a closed convex set of R_+^q. A function W from $C \times R_+$ into R_+ is an <u>aggregator</u> function if it satisfies the following properties :

W1 : continuous ; $W(x,0) \leq M, \forall x \in C$;

W2 : concave ;

for some $\beta \in [0,1[$

W3 : $|W(x,z) - W(x,z')| \leq \beta|z-z'|$,

$\forall x \in C, \forall z, \forall z' \in R_+$;

W4 : $z < z' \Rightarrow W(x,z) \leq W(x,z')$

I.2 - Let X be the space $(R_+^m)^\infty$ endowed with the product topology. An element in X is denoted by $\underset{\sim}{x} = (x_0, x_1 \ldots)$. Let L denote the shift operator on X, i.e $\underset{\sim}{Lx} = (x_1, x_2, \ldots)$.

Let Q be a closed convex, L-invariant subset of X ; S will denote the space of bounded continuous functions from Q into R_+ endowed with the sup norm, $||u|| = \sup_{x \in Q} u(\underset{\sim}{x})$. Let $f_r : X \to (R_+^m)^{r+1}$ be the map $\underset{\sim}{x} \to (x_0, x_1, \ldots, x_r)$ and let $C = f_r(Q)$.

We have the following theorem :

<u>Theorem I.1</u>

With every aggregator W defined on $C \times \mathbb{R}_+$ one can associate an operator on S as follows : $T_W\, u(\underset{\sim}{x}) = W(f_r(\underset{\sim}{x}), u(L\,\underset{\sim}{x}))$.

T_W is a contraction. Hence there exists a unique u, which is concave, such that :

$$\forall \underset{\sim}{x} \in Q,\ u(\underset{\sim}{x}) = W(f_r(\underset{\sim}{x}), u(L\,\underset{\sim}{x})).$$

Proof

Let $T_W\, u(\underset{\sim}{x}) = W(f_r(\underset{\sim}{x}), u(L\,\underset{\sim}{x}))$, with $u \in S$.

$T_W\, u$ belongs to S since it is continuous and bounded on Q ; indeed by W1 and W3:

$$\forall \underset{\sim}{x} \in Q,\ |T_W\, u(\underset{\sim}{x})| \leq \beta |u(L\,\underset{\sim}{x})| + W(f_r(\underset{\sim}{x}),0) \leq \beta ||u|| + M$$

By W3, T_W is a β-contraction on S. The unique fixed point u is concave since under W2 and W4, T_W maps concave functions into themselves.

Q.E.D

Let us introduce the following axioms :

W2 bis : W is concave and for every z, $W(.,z)$ is strictly concave.

W4 bis : $(x,z) \neq (x',z')$ and $(x,z) \leq (x',z')$ implies $W(x,z) < W(x',z')$

W5 : $0 \in C$ and $W(0,0) = 0$

The following propositions is proved in Dana and Le Van [1987].

Proposition I.1

a) If W satisfies W1, W2 bis, W3, W4, then u is strictly concave

b) If W satisfies W1, W2, W3, W4bis, then u is increasing

c) If W satisfies W1 - W5, then $u(0) = 0$

I.3 - Examples

Example I.1 : The discounted case

Let $Q = X$; then $C = (R^m_+)^{r+1}$

Let $v : C \to [0,1]$ be any continuous concave function.

Let $W : C \times R_+ \to R_+$ be defined by

$$W(x_0,x_1,\ldots, x_r ; z) = v(x_0,x_1,\ldots,x_r) + \beta z \ ; \ \beta \in [0,1[.$$

W is an aggregator and the unique fixed point of T_W

is $$u(\underset{\sim}{x}) = \sum_{t=0}^{\infty} \beta^t v(x_t,x_{t+1},\ldots,x_{t+r})$$

Example I.2 : Lucas and Stokey's example [1984]

Let $Q = X$, $r = 0$, $C = R^m_+$. Let $W : C \times R_+ \to R_+$ satisfy W1-W4. Then its unique fixed point satisfies :

$$\forall \underset{\sim}{x} \in Q \ , \ u(\underset{\sim}{x}) = W(x_0,u(L\underset{\sim}{x}))$$

Example I.3 : Beals and Koopmans' example [1969]

Let $W = X$, $r = 0$, $C = R^m_+$. Consider a function

$\omega : [0,1] \times R_+ \to R_+$ that satisfies W1 W4 bis and $v : R_+^m \to [0,1]$ be any concave, strictly increasing continuous function.

Then $W : C \times R_+ \to R_+$ defined by $W(x,z) = \omega(v(x),z)$ is an aggregator and the unique fixed point u of T_W satisfies

$$\forall \underset{\sim}{x} \in Q,\ u(\underset{\sim}{x}) = \omega(v(x_0),\ u(L\,\underset{\sim}{x}))$$

An example considered by Koopmans et ali [1964] is

$$\omega(x,z) = \text{Log}(1+\beta x^\delta + \gamma z)$$

with $\beta, \gamma, \delta > 0,\ \gamma < 1\ ,\ \delta < 1.$

II - A GENERALISED Mc KENZIE MODEL

Consider a quadruple (A, T, C, W) which satisfies the following hypothesis :

A1 : A is a closed convex subset of R_+^p with non empty interior

A2 : T is a set valued continuous correspondence from A into A with non empty compact convex values. Its graph $C = \left\{(x_0,x_1),\ x_0 \in A,\ x_1 \in T(x_0)\right\}$ is closed and convex.

A3 : W is an aggregator defined on $C \times R_+$ and for every fixed $(x_0,\ z)$, the map $W(x_0,.,z)$ is strictly concave

Let $X = (R_+^p)^\infty$ endowed with the product topology.

Let $Q = \left\{\underset{\sim}{x} \in X,\ x_0 \in A,\ x_{t+1} \in T(x_t),\ \forall t \geq 0\right\}$, then Q is a closed convex L-invariant subset of X and $C = f_1(Q)$. It follows from Theorem I.1 that there exists an unique continuous concave function u from Q into R_+ which satisfies $\forall \underset{\sim}{x} \in Q,\ u(\underset{\sim}{x}) = W(x_0,x_1,u(L\,\underset{\sim}{x}))$.

As, $\forall \bar{x}_0 \in A$, the set $X(\bar{x}_0) = \left\{ (\underset{\sim}{x}) \in X,\ x_{t+1} \in T(x_t),\ \forall t \geq 1,\ x_0 = \bar{x}_0 \right\}$ is compact, the following problem $P_{x_0,u}$ has a solution :

$$\max u(\underset{\sim}{x})$$

$$\underset{\sim}{x} \in Q, \text{ and } x_0 \text{ fixed in } A$$

Under A3 the solution is unique and is the trajectory of a dynamical path obtained as follows. Let $V : A \to R_+$ denote the value of $P_{x_0,u}$. We have :

Theorem II.1

a) V satisfies Bellman's equation

$$V(x_0) = \max \left\{ W(x_0, x_1, V(x_1)), x_1 \in T(x_0) \right\} \text{ and is} \qquad (1)$$

concave and continuous

b) $\underset{\sim}{x}$ solves $P_{x_o,u}$ iff $x_t = \varphi^t(x_0)$ with

$$\varphi(x_0) = \text{Arg max} \left\{ W(x_0, x_1, V(x_1)),\ x_1 \in T(x_0) \right\} \qquad (2)$$

Proof : It is omitted since quite standard in dynamic programming.

Q.E.D

Let us now assume :

A2 bis : T satisfies A2, and C has a non empty interior.

We then have :

Proposition II.1

Assume $(x_0, \varphi(x_0))$ is in the interior of C and $W(., x_1, z)$ is differentiable for every fixed (x_1, z). Then V is differentiable at x_0 and one has :

$$V'(x_0) = \frac{\partial W}{\partial x_0}(x_0, \varphi(x_0), V(\varphi(x_0))) \qquad (3)$$

Proof : As $(x_0, \varphi(x_0)) \in \text{int } C$, there exists a neighborhood $N(x_0)$ of x_0 such that $\varphi(x_0) \in T(x)$, $\forall x \in N(x_0)$ (see appendix 1). Then one has $W(x, \varphi(x_0), V(\varphi(x_0))) \leqslant W(x, \varphi(x), V(\varphi(x)))$, $\forall x \in N(x_0)$. Apply Benveniste and Scheikman's [1979] lemma 1 to get the result.

The following result is also straightforward :

Proposition II.2

Let $\{\bar{x}_t\}$ be an optimal solution. Assume W differentiable and $(\bar{x}_t, \bar{x}_{t+1}, V(\bar{x}_{t+1}))$ in the interior of $C \times R_+$, for every $t \geqslant 0$. Then $\{\bar{x}_t\}$ satisfies Euler's equation :

$$\frac{\partial W}{\partial x_1}(\bar{x}_t, \bar{x}_{t+1}, \bar{z}_{t+1}) + \frac{\partial W}{\partial z}(\bar{x}_t, \bar{x}_{t+1}, \bar{z}_{t+1}) \frac{\partial W}{\partial x_0}(\bar{x}_{t+1}, \bar{x}_{t+2}, \bar{z}_{t+2}) = 0 \quad (4)$$

with $\bar{z}_t = V(\bar{x}_t)$, $\forall t \geqslant 0$

□

Let us introduce the following hypothesis :

W6 : W is continuously differentiable

W7 : $\frac{\partial W}{\partial x_{1,j}}(x_0, x_1, z)$ has a constant sign in $C \times R_+$

Remark II.1

In the neoclassical theory of capital with discounting (see example I.1), one usually assumes that $\frac{\partial W}{\partial x_{1,j}}(x_0, x_1, z) = \frac{\partial v}{\partial x_{1,j}}(x_0, x_1)$ is negative for all j. W6 is a less stringent hypothesis since the sign may be positive for some coordinates.

The following theorem is based on Mangasarian's proof [1966] :

Theorem II.2

Assume in addition W6 and W7. Let x_0 be given. Let the unique optimal solution $\{\bar{x}_t\}$ of $P_{x_0,u}$ have the property that if sign $\frac{\partial W}{\partial x_{1,j}}$ equals one then $\{\bar{x}_{t,j}\}$ is bounded. Then any uniformly bounded solution of (4) in Q with initial data x_0 is optimal

Proof Let $\{x_t\}$ be a uniformly bounded solution of (4).

Let $z_0 = W(x_0, x_1, V(x_1))$

$$\bar{z}_0 = W(x_0, \bar{x}_1, V(\bar{x}_1)) = V(x_0)$$

and for $t \geq 1$ $\quad z_t = V(x_t) \geq W(x_t, x_{t+1}, V(x_{t+1}))$, since $x_{t+1} \in T(x_t)$

$$\bar{z}_t = V(\bar{x}_t) = W(\bar{x}_t, \bar{x}_{t+1}, V(\bar{x}_{t+1}))$$

We shall prove that $x_t = \bar{x}_t$, $\forall t \geq 0$

If $x_1 \neq \bar{x}_1$, then by concavity of W

$$0 > z_0 - \bar{z}_0 = W(x_0, x_1, V(x_1)) - W(x_0, \bar{x}_1, V(\bar{x}_1)) \geq \frac{\partial W}{\partial x_1}(x_0, x_1, z_1)\,(x_1 - \bar{x}_1) + \frac{\partial W}{\partial z}(x_0, x_1, z_1)(z_1 - \bar{z}_1)$$

As $z_t \geq W(x_t, x_{t+1}, V(x_{t+1}))$, $\forall t \geq 1$

and $\underset{\sim}{x}$, $\underset{\sim}{z}$ verifie (4), by induction, one gets :

$$\forall t \geq 2,\ 0 > z_0 - \bar{z}_0 \geq \prod_{s=1}^{t-1} \frac{\partial W}{\partial z}(x_{s-1}, x_s, z_s)\ \frac{\partial W}{\partial x_1}(x_{t-1}, x_t, z_t)(x_t - \bar{x}_t)$$

$$+ \prod_{s=1}^{t} \frac{\partial W}{\partial z}(x_{s-1}, x_s, z_s)\ (z_t - \bar{z}_t)$$

As $\underset{\sim}{z}$, $\underset{\sim}{\bar{z}}$, $\underset{\sim}{x}$ are bounded and $\sup_{x_0, x_1, z} \frac{\partial W}{\partial z}(x_0, x_1, z) \leq \beta < 1$

$$z_0 - \bar{z}_0 \geq \lim \sup_t \prod_{s=1}^{t-1} \frac{\partial W}{\partial z}(x_{s-1}, x_s, z_s)\ \frac{\partial W}{\partial x_1}(x_{t-1}, x_t, z_t)(-\bar{x}_t)$$

The R.H.S is non negative since :

- either, sign $\frac{\partial W}{\partial x_{1,j}} = +1$, in which case the sequence $\{x_{t,j}\}$ is bounded and the claim is true as $||\frac{\partial W}{\partial z}|| \leq \beta$ and W is continuously differentiable ;

- either, sign $\frac{\partial W}{\partial x_{1,j}} = -1$ in which case the claim obviously holds.

Hence $z_0 = \bar{z}_0$ or $x_1 = \bar{x}_1$. A similar proof shows by induction that $x_t = \bar{x}_t$, $\forall t \geq 0$.

Q.E.D

As in capital theory, we shall use theorem II.2 to give local convergence results.

Consider now the linearized Euler's equation. Let $\chi_t = (x_t, x_{t+1}, V(x_{t+1}))$

$$\left[\frac{\partial^2 W}{\partial x_1\, \partial x_0} (\chi_t) + \frac{\partial^2 W}{\partial z\, \partial x_0} (\chi_t)\ \frac{\partial W}{\partial x_0} (\chi_{t+1}) \right] dx_t$$

$$+ \left[\frac{\partial^2 W}{\partial x_1^2} (\chi_t) + \left(\frac{\partial^2 W}{\partial x_1\, \partial z} (\chi_t) + \frac{\partial^2 W}{\partial z\, \partial x_1} (\chi_t) + \frac{\partial^2 W}{\partial z^2} (\chi_t)\ \frac{\partial^2 W}{\partial x_0} (\chi_{t+1}) \right) \frac{\partial W}{\partial x_0}(\chi_{t+1}) \right. \tag{5}$$

$$+ \frac{\partial W}{\partial z}(x_t) \frac{\partial^2 W}{\partial x_0^2}(x_{t+1})\Big] dx_{t+1}$$

$$+ \frac{\partial W}{\partial z}(x_t) \left[\frac{\partial^2 W}{\partial x_0 \partial x_1}(x_{t+1}) + \frac{\partial^2 W}{\partial x_0 \partial z}(x_{t+1}) \frac{\partial W}{\partial x_0}(x_{t+2})\right] dx_{t+2} = 0$$

Let us introduce the following hypothesis.

S1 : W fulfills W1, W2 bis, W3, W4

S2 : W is at least three times continuously differentiable

S3 : All solutions of $P_{x_0,u}$ are interior solutions so that Euler's equation is satisfied at any optimal path

S4 : Every steady state x* is a regular zero of Euler's equation ; moreover (x*,. x*) ∈ int C.

$$S5 : \det \left[\frac{\partial^2 W}{\partial x_0 \partial x_1}(x^*) + \frac{\partial^2 W}{\partial x_0 \partial z_1}(x^*) \frac{\partial W}{\partial x_0}(x^*)\right] \neq 0$$

By the implicit function theorem,under S5, (x_{t+2}, x_{t+1}) can be expressed as a C^2-function of (x_{t+1}, x_t) in a neighborhood of (x*, x*). Let us denote by F this mapping. We assume furthermore :

S6 : The Jacobian of F, DF(x*,x*) is a hyperbolic isomorphism of R^{2p} with 2p eigenvalues, $|\lambda_i| < 1$ for $i \leq p$, and $|\lambda_i| > 1$ for $i > p$.

There exists a decomposition of $R^{2p} = E_1 \oplus E_2$ such that $DF(x^*,x^*)(E_i) = E_i$, for i = 1,2. The restriction of DF(x*,x*) to E_1(resp. E_2) has eigenvalues inside (resp. outside) the unit circle. Let us assume :

S7 : "Regularity condition"

The projection of E_1 on $R^p \times \{0\}$ is an isomorphism.

Theorem II.3

Assume S1-S7. Then, every optimal path with initial condition x_0 sufficiently close to a steady state x^* converges to it.

Proof - The argument is Scheinkman's [1976] page 25. Given any x sufficiently close to x^*, one can find by the regularity condition a unique x_1 such that (x_0,x_1) is on the stable manifold at x^*. Since (x^*,x^*) belongs to int C, and the stable manifold is invariant, one can choose x_0 sufficiently near x^* such that the path generated by Euler's equation verifies $x_{t+1} \in T(x_t)$ for every t, and converges to x^*. By Theorem II.2, it is optimal.

Q.E.D

In the one dimensional case, one can generalise the monotonicity results on the optimal trajectory proved by Benhabib and Nishimura [1985] and Benhabib et ali [1985].

Let us introduce the following assumptions

A4.1 $$\left.\begin{array}{l} x_1 \in T(x_0) \\ x_1' \leq x_1 \\ x_0' \geq x_0 \end{array}\right\} \text{ implies } \begin{array}{l} x_1' \in T(x_0) \\ x_1 \in T(x_0') \end{array}$$

A4.2 $$\left.\begin{array}{l} x_1 \in T(x_0) \\ x_1' \geq x_1 \\ x_0' \leq x_0 \end{array}\right\} \text{ implies } \begin{array}{l} x_1' \in T(x_0) \\ x_1 \in T(x_0') \end{array}$$

We have the following result.

Theorem II.4

Let $A = [0,1]$. Assume A2 bis, A3, A4.1 or A4.2 and W continuously differentiable. Assume that for some x_0, $\frac{\partial W}{\partial x_0}(x_0,.,V(.))$ is an increasing (respectively decreasing) function. Then φ is non decreasing (resp. non increasing) in a neighborhood of x_0. If the previous condition holds at every x_0, then every optimal path converges towards a steady state (resp. converges to a steady state or a period two cycle).

Proof Assume that for some x_0, $\frac{\partial W}{\partial x_0}(x_0,\ .,\ V(.))$ is increasing. As A is compact, the condition still holds in a neighborhood of x_0, $N(x_0)$. Let $x_0' \in N(x_0)$, $x_0' > x_0$. Let us denote $x_1 = \varphi(x_0)$, $x_1' = \varphi(x_0')$.

Assume $x_1' < x_1$. If A4.1 holds then $x_1' \in T(x_0)$ and $x_1 \in T(x_0')$ and therefore :

$$W(x_0,x_1,\ V(x_1)) \geq W(x_0,x_1',\ V(x_1'))$$

and $$W(x_0',x_1',\ V(x_1')) \geq W(x_0',x_1,\ V(x_1))$$

Thus

$$W(x_0,x_1,V(x_1)) - W(x_0',x_1,V(x_1)) + W(x_0',x_1',V(x_1')) - W(x_0,x_1',V(x_1')) \geq 0$$

As $x_1' \in T(u)$ and $x_1 \in T(u)$, $\forall u \geq x_0$, this is equivalent to :

$$\int_{x_0}^{x_0'} \left(\frac{\partial W}{\partial x_0}(u,x_1',V(x_1')) - \frac{\partial W}{\partial x_0}(u,x_1,V(x_1))\right) du \geq 0$$

But this quantity is by choice of x_0' strictly negative : a contradiction. Therefore if $x_0' > x_0$, $\varphi(x_0') \geq \varphi(x_0)$.A similar proof may be given if A4.2 is assumed.

The proof of convergence towards a steady state or a period two cycle is a well-known fact of monotonic maps of the interval.

Q.E.D

Remark II.2

When W is linearly separable, i.e $W(x_0,x_1,z) = V(x_0,x_1) + \beta z$ with $0 \leq \beta < 1$, then the condition in theorem II.4 is the well-known condition $\frac{\partial^2 V}{\partial x_0 \partial x_1}(x_0, x_1) > 0$ (resp. < 0).

Next section deals with the main result of the paper. We show that modelling Pareto-optimality in an infinite horizon economy where agents have recursive preferences leads to an optimisation problem which is a particular case of the generalised Mc Kenzie model presented above.

III - A MODEL OF PARETO-OPTIMALITY

III.1 - Notations

i) Throughout this section, we shall use in R^h, where h is an integer, the following notations :

$$z' \geq z \iff \forall j=1, \ldots, h \quad , z_j' \geq z_j$$

$$z' > z \iff z' \geq z \quad \text{and} \quad z' \neq z$$

$$z' >> z \iff z_j' > z_j \; ; \forall j$$

ii) X_h will denote the space $(R_+^h)^\infty$ endowed with the product topology ;

iii) Let n be an integer, x an element of $(R^h)^n$, i.e $x = (x^1, .., x^n)$, where $x^i \in R^h$, $\forall i=1,...,n$.

Then, by definition, $\hat{x} = \sum_{i=1}^{n} x^i$;

(iv) (x^i) will denote $(x^1,...,x^n)$.

Remark III.1 - In appendix 1, we prove that the correspondence Σ :

$$\hat{x} \in R_+^h \to \left\{ (x^1,...,x^n) \in (R_+^h)^n \quad ; \quad \sum_{i=1}^{n} x^i = \hat{x} \right\}$$

is continuous.

III.2 - The economy

We consider an economy with n consumers, each of them lives for an infinite number of periods t = 1,2,...

The economy is described by the list :

$$\mathcal{E} = (X_m \ ; W^i,\ i = 1,...,n \ ; X_p \ ; B \ ; k_0)$$

X_m is the consumption space of each agent. Agent i has utility function $u^i : X_m \to R_+$ defined by an agregator W^i. X_p is the space of sequences of capital. B is the "technology correspondence". It associates with a capital stock k a set of pairs (x,y) of current consumption goods x and next period capital stock y that are jointly producible. k_0 is the initial capital stock. We shall explicit below the assumptions made on the preferences of the agents and the technology.

III.2.1 - Preferences

For every i, W^i is an aggregator function defined on $R_+^m \times R_+$ which satisfies W1, W2 bis, W3, W4 bis, W5, W6.

By theorem I.1 and example I.2, agent i's preferences can be represented by a utility function $u^i : X_m \to R_+$ that verifies :

$$\forall \underset{\sim}{x}^i \in X_m \ , \ u^i(\underset{\sim}{x}^i) = W^i(x_0^i, u^i(L\underset{\sim}{x}^i))$$

It follows from proposition I.1 that u^i is strictly concave increasing and $u^i(0) = 0$.

III.2.2 - Technology

The technology is characterised by a correspondence

$$B : R_+^p \to R_+^m \times R_+^p$$

with the following properties :

B0 : B is continuous

B1 : for each k, B(k) is convex, compact, non empty

B2 : $(x,y) \in B(k)$, and $0 \leq (x',y') \leq (x,y)$ implies $(x',y') \in B(k)$

B3 : $0 \leq k' \leq k$ implies $B(k') \subseteq B(k)$

Define, for $\lambda \in [0,1]$, and $x, x' \in R^m$, $y, y' \in R_+^p$, $k, k' \in R_+^p$

$$x^\lambda = \lambda x + (1-\lambda)x'$$

$$y^\lambda = \lambda y + (1-\lambda)y'$$

$$k^\lambda = \lambda k + (1-\lambda)k'$$

B4 : if $(x,y) \in B(k)$; $(x',y') \in B(k')$, then $(x^\lambda, y^\lambda) \in B(k^\lambda)$

B5 : There exists $x > 0$ such that $(x,y) \in B(0)$

B6 : $k > 0$ implies that there exist $x > 0$, $y > 0$, $(x,y) \in B(k)$

B7 : Let $(x,y,k) \neq (x',y',k')$
If $(x,y) \in B(k)$, $(x',y') \in B(k')$

Then $\forall \lambda \in]0,1[$, there exists $x'' > x^{\lambda}$ such that $(x'', y^{\lambda}) \in B(k^{\lambda})$

B8 : The map $(k,y) \to \left\{ x \mid (x,y) \in B(k) \right\}$ is lower semi-continuous.

Example III.1

Let $F(x,y,k)$ from $R_+^m \times R_+^p \times R_+^p \to R$ be a continuous, strictly convex function, strictly increasing in x and y, strictly decreasing in k, with $F(0,0,0) < 0$. Let $B(k) = \left\{(x,y),\ F(x,y,k) \leq 0\right\}$. Then B satisfies B0-B8.

III.2.3 - Feasible consumption paths. Utility set

A consumption path $(\underset{\sim}{x}^i) \in (X_m)^n$ is feasible from k_0 if it belongs to the set $X(k_0) = \left\{(\underset{\sim}{x}^i) \in (X_m)^n,\ \exists\ \underset{\sim}{k} \in \chi_p,\ (\hat{x}_t, k_{t+1}) \in B(k_t),\ \forall t \geq 0\ ;\ k_0 \text{ given}\right\}$

The utility attainable set from k_0 is defined as follows

$$U(k_0) = \left\{(z_i) \in R_+^n\ ;\ z^i = u^i(\underset{\sim}{x}^i)\ ;\ (\underset{\sim}{x}^i) \in X(k_0)\right\}$$

We have :

Theorem III.1

Assume W1, W2 bis, W3, W4bis, W5 , B0-B6.

a) For every k, U(k) is compact, strictly convex and satisfies free disposal : $\forall u \in U(k)$, $0 \leq u' \leq u$ implies $u' \in U(k)$.

b) For every $\lambda \in [0,1]$, $\lambda\, U(k) + (1-\lambda)\, U(k') \subset U(k^{\lambda})$

c) $\forall k \geq 0$, U(k) has non empty interior

d) The correspondence, denoted by U, from R_+^p into R_+^n : $k \to U(k)$ is continuous

Proof - See Dana and Le Van [1987]

Q.E.D

III.3 - Description of Pareto-optimality

Recall that a consumption path $(\underset{\sim}{x}^i) \in X(k_0)$ is pareto-optimal if there exists no $(\underset{\sim}{x}'^i) \in X(k_0)$ such that $(u^i(\underset{\sim}{x}'^i)) > (u^i(\underset{\sim}{x}^i))$.

Let π^{-1}: $R^n \to R^{n-1}, (z_1, z_2,\dots,z_n) \to (z_2,\dots,z_n)$ and $A = \text{graph } \pi^{-1} \circ U$.

From theorem III.1, A is closed convex and has a non empty interior. Let $\zeta_0 = (k_0, (z_0^i)_{i\geq 2}) \in A$ be given. $(z_0^i)_{i\geq 1}$ is a Pareto-optimal utility vector iff z_0^1 solves :

$$(P) \qquad \max \left\{ z^1 ;\ z^i \geq z_0^i\ ,\ i \geq 2\ ,\ (z^i)_{i\geq 1} \in U(k_0) \right\}$$

Since $U(k_0)$ is compact, (P) has a solution. Let $V(\zeta_0)$ denote the value of this problem. (P) can be rewritten as (P') :

$$\max \left\{ W^1(x_0^1, z_1^1)\ ;\ W^i(x_0^i, z_1^i) \geq z_0^i,\ i \geq 2\ ;\ \exists\ k_1,\ (\hat{x}_0, k_1) \in B(k_0) \text{ and } (z_1^i) \in U(k_1) \right\}$$

Let $\zeta_1 = (k_1, (z_1^i)_{i\geq 2}) \in A$.

Then one has :

$$V(\zeta_0) = \max \left\{ W^1(x_0^1, V(\zeta_1))\ ;\ W^i(x_0^i, z_1^i) \geq z_0^i,\ i \geq 2\ ;\ (\hat{x}_0, k_1) \in B(k_0),\ \zeta_1 \in A \right\}$$

The main purpose of this section is to show that problem (P) is equivalent to a generalised Mc Kenzie model with state space A and characteristics we shall next define.

Consider the following correspondence $T : A \to A$

$$\zeta_0 \to \left\{ \zeta_1 \ ; \ \exists (x^i),\ i \geq 1,\ (\hat{x},k_1) \in B(k_0),\ W^i(x^i,\ z_1^i) \geq z_0^i,\ \forall i \geq 2 \ ; \ \zeta_1 \in A \right\}$$

Let C = graph T as in part two. Clearly C is closed and convex. Define $\Psi : C \to R_+^m$ by

$$\Psi(\zeta_0,\zeta_1) = \left\{ x^1 \in R_+^m \ ; \ \exists\ x^i,\ i \geq 2\ ,\ (\hat{x},k_1) \in B(k_0) \ ; \ W^i(x^i,z_1^i) \geq z_0^i,\ \forall i \geq 2 \right\}$$

Proposition III.1

Assume W1, W2 bis, W3, W4 bis, W5, W6 and B0 - B8.

T and Ψ are continuous, compact convex valued.

Proof - it is given in appendix 2.

Q.E.D

Let us next define $\bar{W} : C \times R_+ \to R_+$ by

$$\bar{W}(\zeta_0,\zeta_1,z) = \max \left\{ W^1(x,z),\ x \in \Psi(\zeta_0,\zeta_1) \right\}$$

We have :

Proposition III.2

a) $\bar{W}$ satisfies W1, W2, W3, W4.

b) Let $\bar{x} = \operatorname{Arg\,max} \left\{ W^1(x,z),\ x \in \Psi(\zeta_0,\ \zeta_1) \right\}$

Then if $\bar{W}$ is differentiable with respect to z,

$$\frac{\partial \bar{W}}{\partial z}(\zeta_0,\zeta_1,z) = \frac{\partial W^1}{\partial z}(\bar{x},z)$$

Proof - It is simple hence omitted.

Q.E.D

By theorem I.1, one can associate with $\bar{W}$ a unique continuous concave function $\bar{U}$ such that

$$\bar{U}(\underset{\sim}{\zeta}) = \bar{W}(\zeta_0,\zeta_1, \bar{U}(L\ \underset{\sim}{\zeta}))$$

Where, $\forall t$, $\zeta_t = (k_t,(z_t^i)_{i\geqslant 2})$

and $\underset{\sim}{\zeta} = (\zeta_0, \zeta_1,\dots, \zeta_t,\dots)$ with $\zeta_{t+1} \in T\ (\zeta_t)$

We can now state the main result of this section :

Theorem III.2

Assume W1, W2 bis, W3, W4 bis, W5, W6 and B0 - B8.

Then problem (P) is equivalent to

$(\bar{P})$ $\max \bar{U}(\underset{\sim}{\zeta}) = \max \bar{W}(\zeta_0, \zeta_1, \bar{U}(L\ \underset{\sim}{\zeta}))$

with $\zeta_t \in T\ (\zeta_{t-1})$ and ζ_0 given.

The value function $\bar{V}(\zeta_0)$ verifies a generalised Bellman's equation :

$$\bar{V}(\zeta_0) = \max \left\{ \bar{W}(\zeta_0,\zeta_1, \bar{V}(\zeta_1)) \ ;\ \zeta_1 \in T(\zeta_0)\right\}$$

Proof - It can be found in appendix three.

Q.E.D

III.4 - Examples

III.4.1 - Benhabib - Jafarey - Nishimura [1985]

We have one consumption good and one capital good and a production funtion :

$$\hat{x} = f(k_0) - k_1$$

where f is three times continuously differentiable on $[0,\infty[$, $f' > 0$, $f'' < 0$.

The n agents have recursive preferences defined by n aggregators

$W^i : R_+ \times R_+ \to R_+$, three times continuously differentiable, which verify, besides assumption W1,W2 bis, W3, W4 bis, W5, W6.

W8 : increasing marginal impatience

$$\forall i, \frac{\partial W^i}{\partial x\, \partial z}\left(1 - \frac{\partial W^i}{\partial z}\right) + \frac{\partial^2 W^i}{\partial z^2} \cdot \frac{\partial W^i}{\partial x} \leqslant 0$$

these partial derivatives are evaluated at any constant path $(c,u^i(c,c...))$.

W9 : normality condition

$$\forall i, \frac{\partial W^i}{\partial z} \frac{\partial^2 W^i}{\partial x^2} - \frac{\partial W^i}{\partial x} \frac{\partial^2 W^i}{\partial x\, \partial z} < 0$$

Define for $i \geqslant 2$, $x_0^i = G^i(z_0^i, z_1^i)$ iff $z_0^i = W^i(x_0^i, z_1^i)$. G^i is three times continuously differentiable, strictly convex, increasing in its first coordinate, decreasing in the second. In this special case, $A \subseteq R_+^n$ and the mapping T (section III.3) is

$$T(\zeta_0) = \left\{ \zeta_1 \;;\; k_1 + \sum_{i \geqslant 2} G^i(z_0^i, z_1^i) \leqslant f(k_0) \right\}$$

It can be checked directly that T has convex compact non empty values and that graph T has a non empty interior and

$$\bar{W}(\zeta_0, \zeta_1, z) = W^1(f(k_0) - k_1 - \sum_{i \geq 2} G^i(z_0^i, z_1^i), z)$$

One can verify that W satisfies S1 and S2. We have.

<u>Corollary III.1</u>

Under the assumptions mentioned above, every regular interior steady state is locally stable.

<u>Proof</u> - It has been shown in Benhabib and ali [1985] that every steady state is a saddle-point, i.e, the linearised Euler's equation admits at this point 2n eigenvalues, n of them are inside, the other ones outside the unit circle. The proof of the "regularity condition" can be found in Dana and Le Van [1987]. The conclusion follows from theorem II.3.

Q.E.D

III.4.2 - <u>Lucas and Stokey [1984]</u> : <u>a two-agent exchange economy</u>

In the model considered, there is one consumption good and two agents.

There is no production and at each date there is an exogeneous supply of consumption good $\bar{x}$. Agents consumptions satisfy $x_t^1 + x_t^2 \leq \bar{x}$ at each date t.

Their preferences are assumed to be represented by aggregators $W^i : R_+ \times R_+ \to R_+$ which verify W1, W2 bis, W3, W4 bis, W5, W6, W9 and are assumed to be twice continuously differentiable.

Let $A = [0, u(\bar{x})]$ with $u(\bar{x})$ unique solution of $u(\bar{x}) = W^2(\bar{x}, u(\bar{x}))$.

Let $\zeta_0 = z_0^2$ and $\zeta_1 = z_1^2$. Define G by $x^2 = G(\zeta_0, \zeta_1)$ iff $\zeta_0 = W^2(x^2, \zeta_1)$. Note that $\frac{\partial G}{\partial \zeta_1} < 0$ and $\frac{\partial G}{\partial \zeta_0} > 0$.

One has in this case $T(\zeta_0) = \left\{\zeta_1,\ G(\zeta_0,\zeta_1) \leqslant \bar{x}\right\}$.

and $\quad \bar{W}(\zeta_0,\zeta_1,z) = W^1(\bar{x} - G(\zeta_0,\zeta_1),\ z)$

T is compact convex non empty valued and has a graph with non empty interior. $\bar{W}$ satisfies W1, W2 bis, W3, W4 and is twice continuously differentiable.

Corollary III.2

Under the above assumptions, every optimal sequence $\left\{z_t^2\right\}$ converges towards a steady state.

Proof - Let $\bar{V}$ denote the value function. By theorem II.4, it suffices to show that $\frac{\partial \bar{W}}{\partial \zeta_0}(\zeta_0,\zeta_1,\ V(\zeta_1))$ is increasing in ζ_1, or equivalently that

$$\Delta(\zeta_0,\zeta_1) = \frac{\partial W^1}{\partial x}(\bar{x}-G(\zeta_0,\zeta_1),\ \bar{V}(\zeta_1))\frac{\partial G}{\partial \zeta_0}(\zeta_0,\zeta_1) \text{ is decreasing in } \zeta_1.$$

Recall that ζ_1 is solution to $\max\limits_{\zeta} W^1(\bar{x}-G(\zeta_0,\zeta),\bar{V}(\zeta))$;

Hence

$$\bar{V}'(\zeta_1) = \frac{\frac{\partial W^1}{\partial x}(\bar{x}-G(\zeta_0,\zeta_1),\ \bar{V}(\zeta_1))\ \frac{\partial G}{\partial \zeta_1}(\zeta_0,\zeta_1)}{\frac{\partial W^1}{\partial z}(\bar{x} - G(\zeta_0,\ \zeta_1),\ \bar{V}(\zeta_1))}$$

$$\frac{\partial \Delta}{\partial \zeta_1}(\zeta_0,\zeta_1) = \frac{\left(-\frac{\partial W^1}{\partial z}\ \frac{\partial^2 W^1}{\partial x^2} + \frac{\partial^2 W^1}{\partial x\, \partial z}\ \frac{\partial W^1}{\partial x}\right)}{\frac{\partial W^1}{\partial z}}\ \frac{\partial G}{\partial \zeta_1}\ \frac{\partial G}{\partial \zeta_0} + \frac{\partial W^1}{\partial x}\ \frac{\left(\frac{\partial^2 W^2}{\partial x^2}\ \frac{\partial W^2}{\partial z} - \frac{\partial^2 W^2}{\partial x\, \partial z}\ \frac{\partial W^2}{\partial x}\right)}{\left(\frac{\partial W^2}{\partial x}\right)^3}$$

The R.H.S is negative, by W9, and

$$\text{since} \quad \frac{\partial G}{\partial \zeta_1} = - \frac{\frac{\partial W^2}{\partial z}}{\frac{\partial W^2}{\partial x}} < 0 \quad \text{and} \quad \frac{\partial G}{\partial \zeta_0} = \frac{1}{\frac{\partial W^2}{\partial x}} > 0$$

Q.E.D

Remark 3.1 - In their paper Lucas and Sfokey assume two conditions (besides the purely technical one s) in order to have a convergence theorem. One is W8 (marginal increasing impatience) the other is W9 (normality condition). We don't assume W8 which implies uniqueness of the steady state and the global convergence of the model towards the unique steady state.

Appendix 1

Lemma A.1.

Let G be a continuous, convex compact valued correspondence form $\mathbb{R}^m$ into $\mathbb{R}^m$. Assume $x^o \in \text{int } G(k^o)$, then there exist a neighbourhood $V(k^o)$ of k^o such that for every k in $V(k^o)$, x^o belongs to $G(k)$.

Proof : Let $x^o \in \text{int } G(k^o)$. Then there exists a ball $B(x^o, \rho)$ centered at x^o and with radius ρ included in $G(k^o)$. Since G is lower semi-continuous, there exists a neighbourhood of k^o such that for every k in $V_1(k^o)$, $G(k) \cap B(x^o, \rho) \neq \Phi$.

Assume that the conclusion is false. Then there exists a sequence k^n, $k^n \to k^o$ such that $G(k^n) \cap B(x^o, \rho) \neq \Phi$ but $x^o \notin G(k^n)$.

Let $\bar{x}^n$ denote the projection of x^o on $G(k^n)$ and let y^n be diametrically opposed to $\bar{x}^n$ in $B(x^o, \rho)$, so that $\bar{x}^n$ is also the projection of y^n on $G(k^n)$. Then, $\| \bar{x}^n - x^o \| = \min \{ \| z - x^o \|, z \in G(k^n) \} \to 0$ as $n \to \infty$.

Thus $\bar{x}^n$ converges to x. On the other hand, let y be a cluster point of the sequence y^n. We have $d(y, G(k^o)) = \lim_n d(y_n, G(k^n)) \geq \rho$ so that $y \notin G(k^o)$. On the contrary by construction $y \in S(x^o, \rho) \subset G(k^o)$ a contradiction.

<u>Lemma A2</u> - Let Σ be the correspondence from R^m_+ into $(R^m_+)^n$ defined as follows : $\Sigma(x) = \left\{(x^i) \in (R^m_+)^n, \sum_{i=1}^{n} x^i = x\right\}$. Then Σ is compact convex valued and continuous.

<u>Proof</u> Σ is trivially compact convex valued and has a closed graph.

Let us show that it is lower semi-continuous : let $\hat{x}$ be fixed in R^m_+. Assume $\hat{x}_1 = \ldots = \hat{x}_l = 0$ and $\hat{x}_{l+1} > 0 \ldots \hat{x}_m > 0$.

Let $(x^i) \in \Sigma(\hat{x})$ ie $\sum_i x^i = \hat{x}$. This implies $x^i_h = 0$, $\forall i$, $\forall h = 1 \ldots l$.

Let $\hat{x}_v \to \hat{x}$. En particular $\hat{x}_{vh} \to 0$ for $h = 1 \ldots l$.

Let $1 \leq h \leq l$, define $x'^i_{vh} = \frac{\hat{x}_{vh}}{n}$ then $x'^i_{vh} \to 0$ and $\sum_i x'^i_{vh} = \hat{x}_{vh}$

Let $h \geq l + 1$, then there exist $j(h)$, $\varepsilon > 0$ such that $x^j_h > \varepsilon$.

There exists v_0 such that $v > v_0$ implies $|\hat{x}_{vh} - \hat{x}_h| < \varepsilon$.

Let $x'^j_{vh} = \hat{x}_{vh} - \hat{x}_h + x^j_h > 0$ and for $i \neq j(h)$, $x'^i_{vh} = x^i_h$.

Then $\sum_i x'^i_{vh} = \hat{x}_{vh}$ and $x'^i_{vh} \to x^i_h$ $\forall i$. □

APPENDIX 2

Proof of Proposition III.1

1. T is compact, convex valued and continuous.

Let us recall: $T : \zeta_0 \in A \rightarrow \{ \zeta_1 \in A;\ \exists (x^i)_{i\geq 1}, (\hat{x}, k_1) \in B(k_o)$ and $\forall\, i \geq 2,\ W^i(x^i, z_1^i) \geq z_o^i \}$

with
$$\zeta_0 = (k_o, (z_o^i)\ i \geq 2)$$
$$\zeta_1 = (k_1, (z_1^i)\ i \geq 2)$$

It can be easily checked that T is upper semi-continuous with non-empty convex compact values. We show now that T is lower semi-continuous.

1.1. Suppose $T(\zeta_0) = \{\zeta_1\}$. In this case T is lower semi-continuous at ζ_0 since T is upper semi-continuous with compact values.

1.2. Suppose that there exist ζ_1, ζ_1', $\zeta_1 \neq \zeta_1'$, in $T(\zeta_0)$. Then one can find x and x' such that:

$$(\hat{x}, k_1) \in B(k_o)$$
$$(\hat{x}', k_1') \in B(k_o)$$
$$\forall\, i \geq 2,\quad W^i(x^i, z_1^i) \geq z_o^i$$
$$W^i(x'^i, z_1'^i) \geq z_o^i$$

1.2.1. $k_1 = k_1'$ (and $z_1 \neq z_1'$).

Let α be fixed in $]0\ 1[$. From the strict convexity of $U(k_1)$, one can find $z_1''(\alpha)$ in int $U(k_1)$ such that:

$$z_1''(\alpha) >> z_1^\alpha\ (= \alpha z_1 + (1-\alpha) z_1')$$

and
$$z_1''(\alpha) \rightarrow z_1 \quad \text{when } \alpha \rightarrow 1.$$

Now, let $k_o^\nu \rightarrow k_o$ and $z_o^\nu \rightarrow z_o$. From the continuity of B and Remark III.1, there exist sequences k_1^ν, x^ν, x'^ν, converging respectively to k_1, x and x', verifying:

$$\forall\, \nu,\quad (\hat{x}^\nu, k_1^\nu) \in B(k_o^\nu)$$
$$(\hat{x}'^\nu, k_1^\nu) \in B(k_o^\nu).$$

One has: $(\hat{x}_\alpha^\nu, k_1^\nu) \in B(k_o^\nu)$, (recall that $x_\alpha = \alpha x + (1-\alpha) x'$)

and $\forall\, j \geq 2,\quad W^j(x_\alpha^j, z''^j_1(\alpha)) > z_o^j$.

From Appendix 1, $z_1''(\alpha) \in U(k_1^\nu)$ for ν large enough.

Summing up, one can say:

with every α in $]0\ 1[$ is associated a $\nu(\alpha)$ such that, for every $\nu \geq \nu(\alpha)$:

$$W^j(x_\alpha^{j\nu}, z''^j_1(\alpha)) > z_o^{j\nu}, \ \forall j \geq 2$$
$$(\hat{x}_\alpha^\nu, k_1^\nu) \in B(k_o^\nu)$$
$$z''_1(\alpha) \in U(k_1^\nu).$$

Moreover one can assume that for $0<\alpha < \alpha' < 1$, $\nu(\alpha') > \nu(\alpha)$. Define $z_1^\nu = z''_1(\alpha^j)$ for $\nu(\alpha^j) \leq \nu < \nu(\alpha^{j+1})$, $\{\alpha^j\}$ being an increasing sequence converging to 1. z_1^ν converges to z_1.

1.2.2. $k_1 \neq k'_1$.

Again, let α be fixed in $]0\ 1[$. There exist sequences (x_t^i), $(k_{1,t})$, (x'^i_t), $(k'_{1,t})$, with $k_{1,1} = k_1$, $k'_{1,1} = k'_1$, and such that:

$$z_1^i = U^i(x_1^i, x_2^i, \ldots, x_t^i, \ldots)$$
$$z'^i_1 = U^i(x'^i_1, x'^i_2, \ldots, x'^i_t, \ldots)$$

$$\forall\ t \geq 1, \quad (\hat{x}_t, k_{1,t+1}) \in B(k_{1,t})$$

$$(\hat{x}'_t, k'_{1,t+1}) \in B(k'_{1,t}).$$

From B4

$$(\hat{x}_t^\alpha, k_{1,t+1}^\alpha) \in B(k_{1,t}^\alpha)$$

and from B7 there exist $\tilde{x}^i(\alpha) > x_1^{i\alpha}$, $\forall i$, with the following property:

$$(\hat{\tilde{x}}(\alpha), k_{1,2}^\alpha) \in B(k_1^\alpha).$$

Then, $\forall i$, $\tilde{z}_1^i(\alpha) = U^i(\tilde{x}^i(\alpha), x_2^{i\alpha}, \ldots x_t^{i\alpha}, \ldots) > U^i(x_1^{i\alpha}, x_2^{i\alpha}, \ldots, x_t^{i\alpha}, \ldots) \geq z_1^{i\alpha}$.

Obviously $\tilde{z}_1(\alpha) \in U(k_1^\alpha)$.

The strict convexity of $U(k_1^\alpha)$ implies that there exists $z''_1(\alpha)$ verifying:

$$z''_1(\alpha) >> z_1^\alpha, \ z''_1(\alpha) \in \text{int } U(k_1^\alpha) \text{ and } z''_1(\alpha) \to z_1 \text{ when } \alpha \to 1.$$

Then $W^j(x_\alpha^j, z''^j_1(\alpha)) > z_o^j, \ \forall j \geq 2.$

Now, let $k_o^\nu, z_o^\nu, k_1^\nu, k'^\nu_1, x^\nu, x'^\nu$, converge respectively to k_o, z_o, k_1, k'_1, x and x', verifying:

$$(\hat{x}^\nu, k_1^\nu) \in B(k_o^\nu)$$

$$(\hat{x}'^\nu, k'^\nu_1) \in B(k_o^\nu).$$

Hence, for ν large enough, $W^j(x_\alpha^{j\nu}, z''^j_1(\alpha)) > z_o^{j\nu}$, $\forall j \geq 2$,

and, by Appendix 1, $z''_1(\alpha) \in U(k_1^{\nu\alpha})$.

With every α in $]0\ 1[$ is associated ${}^{a}\nu(\alpha)$ such that, for any $\nu \geq \nu(\alpha)$:

$$\forall j \geq 2\ ,\ W^j(x^{j\nu}_{\alpha}, z''^{j}_{1}(\alpha)) > z^{j\nu}_{o}$$

$$z''_1(\alpha) \in U(k^{\nu\alpha}_1)$$

$$(\hat{x}^{\nu}_{\alpha}, k^{\nu\alpha}_1) \in B(k^{\nu}_o)$$

Moreover, one can assume $\alpha < \alpha' \Rightarrow \nu(\alpha) < \nu(\alpha')$. Define $z^{\nu}_1 = z''_1(\alpha^j)$, $k''^{\nu}_1 = k^{\nu\alpha^j}_1$ for $\nu(\alpha^j) \leq \nu < \nu(\alpha^{j+1})$, $\{\alpha^j\}$ being an increasing sequence converging to 1. z^{ν}_1 and k''^{ν}_1 converge to z_1 and k_1.

2. <u>Ψ is compact convex valued and continuous.</u>

Ψ is obviously upper semi-continuous and compact convex valued. One has to prove that it is lower semi-continuous.

Let $(\zeta^{\nu}_o, \zeta^{\nu}_1) \in$ graph T, $\zeta^{\nu}_o \to \zeta_o$, $\zeta^{\nu}_1 \to \zeta_1$ and $x^1 \in \Psi(\zeta_o, \zeta_1)$.

There exists $(x^i)_{i\geq 2}$ such that $(\hat{x}, k_1) \in B(k_o)$ and $W^i(x^i, z^i_1) \geq z^i_o$, $\forall i \geq 2$.

From B8 and Remark III.1, there exists $x^{\nu} \to x$ such that $(\hat{x}^{\nu}, k^{\nu}_1) \in B(k^{\nu}_o)$.

2.1. Suppose that $x^1 \neq 0$. Assume that $x^1_m > 0$. Assume also that

$$W^i(x^i, z^i_1) > z^i_o \quad \text{for } i = 2,..,p < n$$

and

$$W^i(x^i, z^i_1) = z^i_o \quad \text{for } i > p.$$

Define: $\quad x'^{i\nu} = x^{i\nu}$ for $i = 2,..,p$

then $\quad W^i(x'^{i\nu}, z^{i\nu}_1) > z^{i\nu}_o$ for ν large enough.

For $i > p$ define: $\quad x'^{i\nu}_j = x^{i\nu}_j$ if $j \neq m$

and $x'^{i\nu}_m$ by $\quad W^i(x'^{i\nu}, z^{i\nu}_1) = z^{i\nu}_o$ if $W^i(x^{i\nu}, z^{i\nu}_1) < z^{i\nu}_o$

$$x'^{i\nu}_m = x^{i\nu}_m \quad \text{if } W^i(x^{i\nu}, z^{i\nu}_1) \geq z^{i\nu}_o .$$

Clearly $\quad x'^{i\nu}_m \to x^i_m$.

Define $\quad x'^{1\nu} = \hat{x}^{\nu} - \sum_{i\geq 2} x'^{i\nu}$

then one easily checks that:

$$(\hat{x}'^{\nu}, k^{\nu}_1) = (\hat{x}^{\nu}, k^{\nu}_1) \in B(k^{\nu}_o)$$

$$W^i(x'^{i\nu}, z^{i\nu}_1) \geq z^{i\nu}_o \quad , \forall i \geq 2$$

$$x'^{1\nu} \geq 0 \quad \text{for } \nu \text{ sufficiently large}$$

$$x'^{1\nu} \to x^1 .$$

2.2. $x^1 = 0$.

Since $(\zeta_o^\nu, \zeta_1^\nu) \in$ graph T, there exists x''^ν such that:

$$(\hat{x}''^\nu, k_1^\nu) \in B(k_o^\nu)$$

$$W^i(x''^{i\nu}, z_1^{i\nu}) \geq z_o^{i\nu} \quad , \forall i \geq 2.$$

Define $\quad x'^{i\nu} = x''^{i\nu}$ for $i \geq 2$

$$x'^{1\nu} = 0 \; .$$

One has: $\quad \hat{x}'^\nu \leq \hat{x}''^\nu$

which implies $\quad (\hat{x}'^\nu, k_1^\nu) \in B(k_o^\nu)$.

In other words $\quad 0 \in \Psi(\zeta_o^\nu, \zeta_1^\nu)$ for every ν .

Q.E.D.

APPENDIX 3

Proof of Theorem III.2

Proof - Let $\bar{z}_0^1$, $\bar{\bar{z}}_0^1$ be the values of (P) and ($\bar{P}$). Obviously $\bar{\bar{z}}_0^1 \leqslant \bar{z}_0^1$. We have just to show that $\bar{z}_0^1 \leqslant \bar{\bar{z}}_0^1$. Let $\bar{\underset{\sim}{x}}^1$ be an optimal consumption path.

Denote

$$\bar{z}_t^1 = u^1(L^t \bar{\underset{\sim}{x}}) = W^1 (\bar{x}_t^1, \bar{z}_{t+1}^1)$$

With these paths, one can associate a capital path $\bar{\underset{\sim}{k}}$ and consumption paths $\bar{\underset{\sim}{x}}^i$, for $i \geqslant 2$ of the other agents. Let $\bar{\underset{\sim}{z}}^i$, for $i \geqslant 2$ denote

$$\bar{z}_t^i = u^i(L^t \bar{\underset{\sim}{x}}^i)$$

These paths verify :

$$\forall t \quad (\bar{x}_t, \bar{k}_{t+1}) \in B(\bar{k}_t)$$

$$k_0 \text{ given}$$

for $i \geqslant 2$:

$$\bar{z}_t^i = W^i(\bar{x}_t^i, \bar{z}_{t+1}^i) \text{ .}$$

One has

$$\bar{z}_0^1 \leqslant \max \quad W^1(x_0^1, \bar{z}_1^1) \leqslant \bar{W} (\zeta_0, \bar{\zeta}_1, \bar{z}_1^1)$$

$$(\hat{x}, \bar{k}_1) \in B(k_0)$$

$$(\bar{z}_1^i)_{i \geqslant 1} \in U(\bar{k}_1)$$

By the same way :

$$\bar{z}_1^1 \leqslant \bar{W} (\bar{\zeta}_1, \bar{\zeta}_2, \bar{z}_2^1)$$

Since $\bar{W}$ is non decreasing with respect to z, one gets

$$\bar{z}_0^1 \leq \bar{W}(\zeta_0, \bar{\zeta}_1, \bar{W}(\bar{\zeta}_1, \bar{\zeta}_2, \bar{z}_2^1))$$

by induction

$$\bar{z}_0^1 \leq \bar{U}\,(\underset{\sim}{\bar{\zeta}}) \quad \leq \bar{\bar{z}}_0^1$$

$$\bar{\zeta}_{t+1} \in T\,(\bar{\zeta}_t)$$

$$\zeta_0 \text{ given}$$

Q.E.D

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