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MONOPOLY, SYNDICATE AND SHAPLEY VALUE : ABOUT SOME CONJECTURES*

by

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MONOPOLE SYNDICAT ET VALEUR DE SHAPLEY

A PROPOS DE QUELQUES CONJECTURES

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par Roger GUESNERIE

Au cours des années récentes, tout un courant de la recherche, inspiré par des théoriciens des jeux, s'est attaché à l'explication du "pouvoir de marchandage" que sa taille donne à un agent dans une négociation, avec l'espoir, sans doute trop hâtif, de bâtir une théorie économique du "pouvoir " des syndicats ou des monopoles.

En fait, étant donné un concept de solution en théorie des jeux - c'est-à-dire une hypothèse sur le mécanisme de négociation et sur le type d'issue que le jeu connaîtra - deux concepts de stabilité des "syndicats" peuvent être définis.

. Pour le premier, le syndicat doit être avantageux dans le sens où ses membres reçoivent plus quand ils sont syndiqués que quand ils sont tous non-syndiqués ; (A-Stabilité)

. Pour le second, un joueur syndiqué doit recevoir plus qu'un joueur semblable qui est non-syndiqué, (B-Stabilité)

Avec le concept de solution le plus classique, le coeur, ces deux types de stabilité ont été étudiés dans le cadre d'une économie d'échange à agents nombreux. Les conclusions largement négatives obtenues sur la stabilité des syndicats et des monopoles ont conduit AUMANN à affirmer que "le concept de coeur n'est pas le véhicule convenable à l'explication de l'avantage du monopoleur" et à émettre la conjecture que le concept de valeur de SHAPLEY étant mieux adapté à l'analyse du "pouvoir de marchandage" des agents de "grande taille".

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L'objet de cette note est d'étudier et de réfuter cette conjecture.

Dans une première partie, on présente le problème et on définit rigoureusement les concepts de stabilité de syndicats - relativement à un concept de solution de théorie des jeux - évoqués ci-dessus.

Une seconde partie est consacrée à l'exposé de résultats préliminaires sur le calcul de la valeur de SHAPLEY dans des économies d'échanges dont les agents ont des préférences homothétiques.

La 3ème partie envisage le premier concept de stabilité (A.Stabilité). On y exhibe une classe d'économies à agents nombreux dans laquelle il est avantageux (au sens de la A.Stabilité) de se syndiquer. Cependant on montre que des modifications des préférences convenablement choisies conduiraient à une conclusion inverse. Ce dernier exemple met en évidence certaines propriétés mal connues de la valeur de SHAPLEY (disparitions de "l'ordinalité" qui vaut pour des économies à agents nombreux sans atomes).

La quatrième partie concentre l'attention sur le second concept de stabilité : on donne un exemple numérique d'une économie où les agents ont intérêt à se comporter en resquilleurs vis-à-vis du syndicat dès lors que celui-ci existe.

Quelques énoncés " positifs" suggérés par l'analyse précédente sont démontrés en annexe dans le cadre d'un espace mesuré d'agents avec atomes.

Il apparaît en conclusion que les exemples considérés n'ont rien de pathologiques et que le concept de valeur de SHAPLEY ne prend en compte que très imparfaitement le pouvoir de "menace" qui détermine la force dans la négociation d'un agent de "grande taille" et rend la collusion bénéfique.

I - INTRODUCTION : CONCEPTS OF STABILITY OF SYNDICATES.

Observations of real situations suggest that the strength of an agent in a negociation process increases with its size. Especially, it is commonly believed that binding agreements between a group of agents do improve their bargaining power and are advantageous for the agents of the group. For giving a theoretical basis to such a conjecture, several kinds of approaches can be considered. One of the most attractive consists in trying to find a rationale for such phenomena in modern game theory. The abstract modelling of situations involving conflicts and cooperation and the conceptual tools developed in this fiels, seem appropriate for a theoretical analysis of the influence of binding agreements on the outcome of bargaining processes.

In this perspective, recent works on economic games in the framework of a measure space of economic agents focused the attention on the existence of atoms (see SHITOVITZ [1973], DREZE and Others [1972]). Such "atoms" can be interpreted as resulting from binding agreements between a group of agents and, for example, can be considered as providing an abstract formalization of syndicates of traders (cf. GABZEWICZ and DREZE [1971]). This article refers to this current of research. Following GABZEWICZ-DREZE [1971], we will term a syndicate a group of identical agents —i.e., in the context of an exchange economy a group of agents, all of them having the same initial endowments and the same preferences— who agree that "no proper subset of them will form a coalition with traders outside the group, so that only the group as a whole will enter into broader coalitions". A monopoly will be a special syndicate (which can reduce to a single agent) owning the total quantity available in the economy of a given commodity, a definition conforming to that of AUMANN [1973].

The study of syndicates or monopolies so defined raises the question of their stability, i.e., of the willingness of the members of the syndicate to stay in the syndicate and it is natural to suppose that such a stability will depend upon the advantages the syndicate gives to its members. Actually two comparisons seem relevant to evaluate the advantages of a syndicate as perceived by one of its members. The first one is the comparison of the utility of this member associated with the outcome of the game when there is a syndicate, and his utility when there is no syndicate : this is AUMANN's notion of advantage [1973] . The second one is the comparison of the player's utility when he is syndicated with the utility of an unorganized agent which has the same resources and the same preferences as he has ; the notion of marginal stability of GABZEWICZ-DREZE [1971] is based on such a comparison.

Let us give a more formal definition of these two notions. For that, let us consider a N players cooperative game on which for the sake of simplicity the following assumptions are made[1] :

- Outcomes of the game are représented by a N uple of vectors of $\mathbb{R}^{\ell}$ and are denoted $x = (x_1, \ldots, x_N)$ $(x \in \mathbb{R}^{\ell N})$
- Player i is only concerned by the i^{th} vector of x, x_i $(x_i \in \mathbb{R}^{\ell})$. Preferences of player i are represented by a preordering defined on an appropriate subset of $\mathbb{R}^{\ell}$ and denoted $\underset{\sim}{\alpha}_i$.
- A characteristic set function defines the possibilities of the 2^N admissible coalitions, by associating to each coalition S a subset $V(S) \subset \mathbb{R}^{\ell |S|}$ (with usual notations).

For such a game different solution concepts may be considered. Each of them rest upon "equilibrium" notions or reflects ideas about the way the game is played, the bargaining power of the agents and defines accordingly "reasonable" outcomes for the game. More precisely, an appropriate solution concept allows to select a subset of outcomes which are "solutions" of the game —subset which may be empty, have one or several elements—. Let us call X(SC,.) this subset, which depends upon the solution concept considered SC, and all characteristics of game (V, and $\underset{\sim}{\alpha}_i$) denoted (.) .
Now let us suppose that a subset J of the N players consists in identical agents (identical preferences and symetrical role in the game) and that

a subset $I \subset J$ decides to constitute a syndicate. As a consequence, a new game is played which has formally —since the syndicate acts as a single agent— $N-|I|+1$ players. Supposing that the gains of the syndicate are equally shared between its members, —a natural assumption for identical players—, the preferences of the syndicate as a player can be consistently defined.

The set of admissible coalitions is restricted —since coalitions with some, but not all, players of J are ruled out— but the characteristic function set can be straightforwardly redefined for the new game from the original one.

Finally in the new game, a given solution concept still allows to exhibit a subset of "solutions", for the $N-|I|+1$ players and taking into account the distribution rule between the members of the syndicate, for the N original players.

Let us call XS(SC,I,.) the corresponding subset of outcomes in $\mathbb{R}^{\ell N}$, which depends upon the solution concept considered SC, the syndicate I and the others characteristics of the game.

We are now in position to give formal definitions of the concepts of stability of syndicates —evoked above— with respect to a given solution concept SC.

I is strongly A-stable if $\forall\, x \in X(SC,.)$, $\forall\, x' \in XS(SC,I,.)$ $x'_i \succsim_i x_i$, $\forall\, i \in I$ and $\exists$ a couple (x,x') s.t. $x'_i \succ_i x_i$, $\forall\, i \in I$.

I is weakly A-stable if $\forall\, x \in X(SC,.)$, $\exists\, x' \in XS(SC,I,.)$ s.t. $x'_i \succ_i x_i$, $\forall\, i \in I$.

When X(SC,.) and XS(SC,I,.) reduce to a single point, the two concepts coincide and define what we will term <u>A-stability</u>.

I will be said <u>strongly B-stable</u> if $\forall\, x \in XS(SC,I,.)$, $x_i \succ_i x_j$, $\forall\, i \in I$, $j \in J/I$.

I will be said <u>weakly B-stable</u> if $\exists\, x \in XS(SC,I,.)$ and $j \in J/I$ s.t. $x_i \succ_i x_j$, $\forall\, i \in I$.

When XS(SC) reduces to a point and when all agents of J/I are equally treated by the solution concept SC, the two concepts coincide and one will speak of <u>B-stability</u>.

A-stability refers to the AUMANN's notion of advantage —comparison of the situations of the syndicate members when there is, and when there is no syndicate— when B-stability is related to the marginal stability of GABZEWICZ-DREZE— comparison of the situation of syndicated and non syndicated people of the same type, when there is a syndicate.

The stability is said to be strong, when the syndicate is necessarily advantageous from the point of view adopted (A or B-stability). Weak stability expresses the idea that from the adopted point of view, the syndicate may be advantageous.

The reader will, notice that several alternative formulations of this last idea could have been possible, both for weak A and weak B stability and that the definitions chosen are to some extend arbitrary (2).

Let us also notice that these definitions are relative to games with a finite number of players but could easily be extended to games with a measure space of agents (3) (4).

As we have already noticed, the stability of syndicates has been studied for syndicates of traders in an exchange economy in the framework of a measure space of economic agents, with the solution concept of core.

The main results obtained can be summarized as follows : (supposing that the concepts defined above be rephrased in order to apply to games with an infinite number of players).

1. $XS(Core,I) \supset X(Core) = \{\text{set of competitive equilibria}\}$. Characteristics of allocations belonging to XS have been given by SHITOVITZ [1973] (5).

2a. There exist exchange economies in which syndicates are strongly A-stable. See SHITOVITZ [1973], GABZEWICZ-DREZE [1971].

2b. But cases "without pathological features" can be found in which a syndicate is not even weakly A-stable. See AUMANN [1973].

3. Weak B-stability (and a fortiori strong) cannot "generally" be expected (GABZEWICZ-DREZE [1971]).

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So according to 2, advantageous (2a) as well as disadvantageous (2b) syndicates (in AUMANN's sense) can be found. However, AUMANN notes that "one is almost forced to the conclusion that monopolies which are not particularly advantageous are probably the rule rather than the exception". Such a conclusion, joined to point 3, emphasizes that the analysis fails to reflect the advantages that are supposed to be attached with monopolies or syndicates. This can be seen as a consequence of the inadequacy of the concept of solution of core, an opinion supported by AUMANN, according to whom "the game theoretic notion of core is not the proper vehicle for the explanation...(of the monopolist advantage)". "The (monopolistic) strenght lies in his possibilities, in the bargaining power engendered by the harm he can cause by refusing to trade "a phenomena which is "not foreign to game theory and closely related to the ideas underlying the Shapley value".

The purpose of this paper is to examine this latter conjecture by discussing A- and B-stability for syndicates of traders in an exchange economy when the allocation of commodities is governed by the Shapley value.

Our arguments will be developed by considering limit of replica economies rather than in the framework of an atomic measure space of economic agents—in which a Shapley value can be defined along the lines of HART's analysis [1973]—. The reasons for this choice are mainly twofold. First, HART's article does not provide a direct way of computing the Shapley value of atomic players and does not allow —at least without preliminary work— the study of B-stability. Second, we will deal essentially with examples for which the inconveniences of technalities required for proving existence in a measure space are greater than the advantages of simpler computations of Shapley value in such a context (6).

Thus, our exploration of A and B-stability of syndicates with respect to the Shapley-value —based on the study of specific examples— will proceed as follows :

After giving preliminary results in the second paragraph, we will exhibit in the first part of third paragraph a class of <u>two goods economies</u> for which, at the limit in the n-replica, a syndicate of traders (which is also a monopoly) is <u>strongly A-stable</u>.

In the second part of the third paragraph, we will see how small modifications of the previous example allow us to exhibit syndicates which are not weakly A-stable. Moreover, this phenomenon occurs in a subclass of economies where syndicates are strongly A-stable with respect to the core concept.

In the fourth paragraph an example of absence of B-stability will be presented.

As the reader will see, the examples of failures of A- and B-stability have no pathological features. They provide a disappointing insight on the conjecture according to which the Shapley value would be an appropriate concept for capturing the bargaining power of syndicates or monopolies (even if some "positive" statements are suggested by the examples of paragraph 3).

The implications of the whole study on the directions of future research will be briefly discussed in conclusion.

II - PRELIMINARY RESULTS.

A/ Shapley value in "homogeneous markets" :

The concept of Shapley value defined first in games with transferable utility has been extended by Shapley to games without transferable utility (Shapley [1969]). Thus, Shapley value allocations can be defined in general exchange economies in which there does not exist a money-commodity allowing transfer of utility.

Let $E = (X_i, u_i, w_i)$ be such an economy. Given a vector $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ belonging to S^n, simplex of $\mathbb{R}^n$, ($S^n = \{\lambda \mid \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0\}$) one can associate with the economy the game with side-payments G_λ defined by the following characteristic function :

$$v(\lambda,S) = \text{Max} \sum_{i\in S} \lambda_i u_i(x_i) \quad , \quad \sum_{i\in S} x_i \leq \sum_{i\in S} w_i \quad .$$

In particular

$$v(\lambda,N) = \text{Max} \sum_{i\in N} \lambda_i u_i(x_i) \quad , \quad \sum_{i\in N} x_i \leq \sum_{i\in N} w_i \quad .$$

The Shapley value assignment of this side-payments game is given by the following formula :

$$Sh_i(\lambda) = \frac{1}{n!} \sum_{\omega\in\Omega} v(\lambda, S_{\omega,i} \cup \{i\}) - v(\lambda, S_{\omega,i}) \qquad (1)$$

where Ω is the set of all orderings of the n agents and $S_{\omega,i}$ is the coalition of players preceding player $\{i\}$ in the ordering $\{\omega\}$.

Given a game G_λ, the sequence of utility levels $(\ldots, Sh_i(\lambda)/\lambda_i, \ldots)$[7] corresponding to the Shapley value of the game cannot "generally" be obtained by a pure reallocation of commodities. The Shapley value of the economy is associated with those λ^* (if any) such that the solution of the games G_{λ^*} does not imply any transfer of utility. Precisely a Shapley value allocation is a sequence of bundles of $\mathbb{R}^\ell$ (the commodity space) $(x_1^*, \ldots, x_i^*, \ldots, x_n^*)$ associated with a vector $\lambda^* (\lambda^* \in \mathbb{R}^n)$ such that[8] :

$$\lambda_i^* u_i(x_i^*) = Sh_i(\lambda^*) \quad , \quad \forall\, i = 1, \ldots, n$$

$$\sum_{i\in N} x_i^* \leq \sum_{i\in N} w_i \quad . \qquad (2)$$

Let us now consider an exchange economy E with n agents which have identical preferences. Each of them has a preference preordering defined on $\mathbb{R}_+^\ell$ which can be represented by a utility function u, quasi-concave and homogeneous of degree one and such that $u(0) = 0$. The vector of initial endowments in the economy is $w >> 0$. One can prove

<u>Lemma</u> : In an economy as defined above,

(a) The set of feasible utilities (i.e., the set of vectors $u = (u_1, \ldots, u_n)$ such that $\exists\, (x_1, \ldots, x_n)$ with $u_1 = u(x_1) \ldots u_n = u(x_n)$ and $\sum_1^n x_i \leq w$) is limited by the hyperplanes, $\sum_{i=1}^n u_i \leq u(w)$, $u_i \geq 0$, $\forall\, i$.

(b) When u is strictly increasing and when all initial endowments w_i are nonzero, every Shapley value allocation is associated with equal weights for all the agents ($\lambda^* = (1/n, \ldots, 1/n, \ldots, 1/n)$).

(a) is a well-know property and we will give only an outline of proof. Let us consider $(\bar{x}_1,\dots,\bar{x}_n)$ a Pareto optimal allocation of E. One can find positive numbers $t_1,\dots,t_n$, $t_1 \geq 0$, $t_n \geq 0$, $t_1 + \dots + t_n = 1$, such that $\bar{x}_i \underset{i}{\sim} t_i\, \omega$, $\forall$ i. (Consider the price system p associated with the Pareto optimum, an i s.t. $\bar{x}_i \neq 0$ and the set $\chi_i = \{x'_i \in X_i \mid p.\bar{x}_i = p.x'_i$ and $\bar{x}_i \sim x'_i\}$. It is easy to see that the smallest cone of vertex 0 containing χ_i is convex and must contain χ_j , $\forall$ j $\neq$ i , and must also contain w.) Then if $\bar{u}_i = u(\bar{x}_i)$, $\sum_{i=1}^{n} \bar{u}_i = u(w)$.

(b) Let $(\dots x_i^* \dots)$ be a Shapley value allocation and $\lambda^* = (\dots\lambda_i^*\dots)$ the set of associated weights. Let us suppose that there exists i, j such that $\lambda_i^* > \lambda_j^*$. It follows from the above definitions and (a) that $u_j^* (= u(x_j^*)) = 0$, and $\lambda_j^*\ u_j^* = 0$. Let us consider, then, formulas (1) and (2) above One has :

$$\lambda_j^*\ u_j^* \;=\; Sh_j(\lambda^*)$$

$$Sh_j(\lambda^*) \;\geq\; \lambda_i^*\ u(w_i + w_j) - \lambda_j^*\ u(w_j) \quad ,$$

the right hand side term being a mincrant of the contribution of agent j to the coalition formed by himself and agent i. This term is itself at least equal to $\lambda_i^*(u(w_i + w_j) - u(w_j))$ which according to our assumptions is strictly positive. This is a contradiction. Q.E.D.

This result can be summarized as follows : in a "homogeneous market", i.e., an exchange economy where all agents have an identical utility function u homogeneous of degree one and concave, the Shapley value allocation is unique and the levels of utilities of the agents associated with this allocation are the Shapley value assignments of the game (with transferable utility) whose characteristic function is defined by $v(S) = u(\sum_{i\in S} w_i)$.

B/ <u>Shapley value in replica of homogeneous markets with a syndicate.</u>

Let us consider an initial homogeneous markets E with only two agents, $A(\mathbb{R}_+^{\ell}, u, w_A \neq 0)$, $B(\mathbb{R}_+^{\ell}, u, w_B \neq 0)$ (u is a homogeneous of degree one, strictly increasing concave utility function).

In the n-replica of this economy $E^{(n)}$, all agents of type A form a syndicate of traders in GABZEWICZ-DREZE's sense. The "syndicate" action obeys two conditions : first, agents of type A cannot join separately coalitions : in the process of formation of coalitions the syndicate acts as a single agent ; second, the bundle of goods obtained by the syndicate is equally shared between its members. It follows that if the bundle of goods obtained by the syndicate is x, the total utility of the syndicate members if $(nu(x/n) = u(x)$. Thus, in the following the syndicate can be considered a single agent with the same utility function as the other agents. So the economy $E^{(n)}$ with a syndicate can be considered as having (n+1) agents which can be numbered as follows :

Agent n° 1 is the syndicate : $(\mathbb{R}_+^{\ell}, u, w_1 = nw_A)$.

Agent n° 2 to n+1 are agents of type B : $(\mathbb{R}_+^{\ell}, u, w_i = w_B)$, $i = 2,\ldots,n+1$.

In order to compute the Shapley value of this economy $E^{(n)}$ (which is a homogeneous market in the sense of 2.A,let us consider the set of orderings of the (n+1) agents. This set can be divided into n+1 subsets according to the rank of the syndicate (player 1) in the ordering.

1st subset : player 1 is first in the ordering .

. . . .

p-th subset : player 1 is p-th in the ordering .

. . . .

There are n! orderings in each subset. Let us call $\Delta v(A_p)$ the contribution of player 1 to a coalition of (p - 1) agents of type B (corresponding to the p-th type of ordering).

$Sh_n(1)$, the utility of player 1 corresponding to his Shapley value allocation in the economy $E^{(n)}$ with syndicate can be written down :

$$Sh_n(1) = \frac{1}{(n+1)!} \sum_{p=1}^{p=n+1} n!\, \Delta v(A_p) = \frac{1}{n+1} \sum_{p=1}^{p=n+1} \Delta v(A_p)$$

where according to section 2.A.

$$\Delta v(A_{p+1}) = u(nw_A + pw_B) - u(pw_B) \; .$$

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Hence

$$Sh_n(1) = \frac{1}{n+1} \left[\sum_{p=1}^{p=n+1} u(nw_A + (p-1)w_B) - \sum_{p=1}^{p=n+1} [u(p-1)w_B)\right] .$$

In fact, if we favor the interpretation in terms of "syndicate" we are rather more interested in $(1/n)Sh_n(1)$

$$\frac{1}{n} Sh_n(1) = \frac{1}{n+1} \sum_{p=0}^{p=n} [u(w_A + \frac{p}{n} w_B) - u(\frac{p}{n} w_B)] .$$

Let us consider the limit of this expression when n increases to infinity : one has

$$\frac{1}{n} \sum_{p=0}^{p=n} [u(w_A + \frac{p}{n} w_B)] \geq \int_0^1 [u(w_A + tw_B)\ dt] \quad (9)$$

and

$$\lim \frac{1}{n} \sum(\ ) = \int_0^1 u(w_A + tw_B)\ dt$$

so that

$$\boxed{\lim \frac{1}{n} Sh_n(1) = \int_0^1 u(w_A + tw_B)\ dt - \int_0^1 u(tw_B)\ dt} . \quad (3)$$

III - A-STABILITY.

A/ A subclass of homogeneous markets in which the syndicate is strongly A-stable.

Let us specify economy $E^{(1)}$ of the preceding section 2.B as follows :

$\ell = 2$, the consumption set is $\mathbb{R}^2_+$.

If $x = \binom{\lambda}{\mu}$, $u(x) = \lambda^\alpha \mu^{1-\alpha}$.

$w_A = \begin{pmatrix} w_A \\ 0 \end{pmatrix}$, $w_B = \begin{pmatrix} 0 \\ w_B \end{pmatrix}$.

There are two commodities, the utility function is a COBB-DOUGLAS function, consumer A owns only commodity 1 and consumer B owns only commodity 2. The limit level of utility given to the members of the syndicate by the Shapley value solution, can now be explicitly computed from the above formula.

$$\lim \frac{1}{n} Sh_n(1) = w_A^{\alpha} w_B^{1-\alpha} \int_0^1 t^{1-\alpha} dt = \frac{w_A^{\alpha} w_B^{1-\alpha}}{2-\alpha} .$$

(Here, and in the following, w_A , w_B are numbers and not vectors.)

The limit Shapley value bundle attributed to the members of the syndicate is denoted $\bar{x}_A^{\infty}(\alpha)$ (α being the exponent of the COBB-DOUGLAS function) and

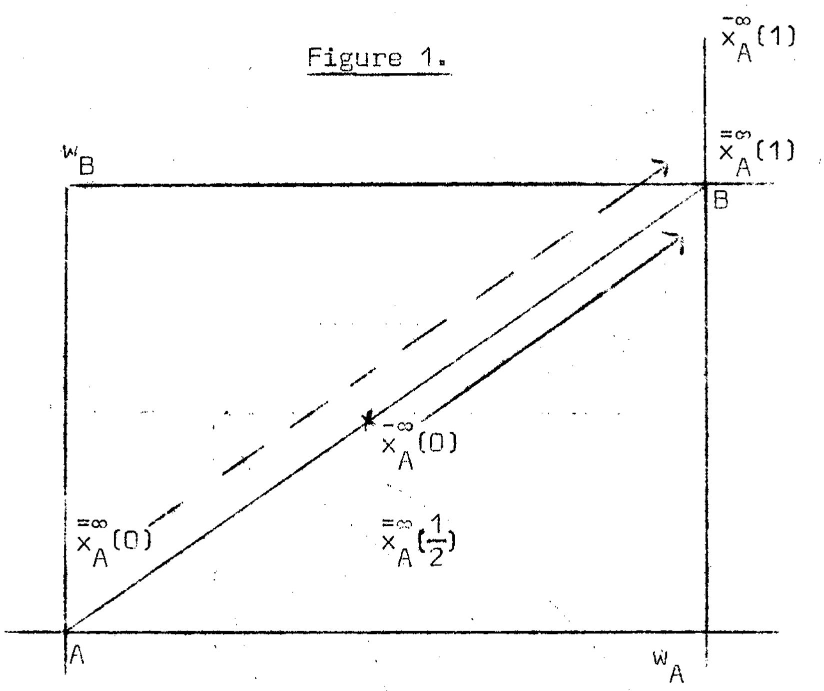
$$\bar{x}_A^{\infty}(\alpha) = \frac{1}{2-\alpha} \begin{pmatrix} w_A \\ w_B \end{pmatrix} .$$

We are now ready to focus attention on A-stability. At this point, two premilinary remarks must be made :

Every replica economy $E^{(n)}$, with or without syndicate is a homogeneous market in the sense of I and consequently has a unique Shapley value allocation. It turns out that the concepts of strong A-stability and of weak A-stability coincide in these replica.

The unique Shapley value allocation of the replica economy $E^{(n)}$ without syndicate converges towards the (unique) competitive equilibrium when $n \to +\infty$[10]. Thus, A-stability of the syndicate for n large enough depends only upon the comparison of the competitive bundle going to the syndicate members and of $\bar{x}_A^{\infty}$, defined above.

Figure 1.



The competitive bundle of a syndicate member, denoted by $\bar{\bar{x}}_A^{\infty}(\alpha)$, can be easily computed when the exponent of the COBB-DOUGLAS function is α

$$\bar{\bar{x}}_A^{\infty}(\alpha) = \alpha \begin{pmatrix} w_A \\ w_B \end{pmatrix} .$$

$\bar{x}_A^{\infty}(\alpha)$, $\bar{\bar{x}}_A^{\infty}(\alpha)$ can be visualized on an EDGEWORTH diagram (Figure 1). As

$$\alpha < \frac{1}{2-\alpha} \quad , \qquad \forall\, \alpha \in [0,1[$$

the Shapley value bundle $\bar{x}_A^{\infty}(\alpha)$ is strictly preferred by the members of the syndicate to the competitive bundle $\bar{\bar{x}}_A^{\infty}(\alpha)$, $\alpha \in [0,1[$.

One can summarize this :

Let us consider the subclass of homogeneous economies (with two goods, two agents, and COBB-DOUGLAS utility functions with $0 \leq \alpha < 1$), defined here. In this subclass of economies, the syndicate constituted by all agents of one type (the agents of the other type remaining unorganized), is (strongly) A-stable in replica economies $E^{(n)}$, as soon as n is large enough.

B/ An example of a syndicate not weakly A-stable.

Let us come back to the preceding example by considering the function $\phi : \mathbb{R}^+ \overset{\phi}{\to} \mathbb{R}^+$ such that $\phi(\lambda) = w_A^{\alpha}\, \lambda^{1-\alpha}$. ϕ is nothing else than the section of the utility function $u(\mu,\lambda)$ by the plane $\mu = w_A$. Let us draw the graph of ϕ for $\alpha = 2/3$ (Figure 2). According to formula II.B(3), the limit Shapley value of the syndicate is the surface of the shaded area times $1/w_B$.

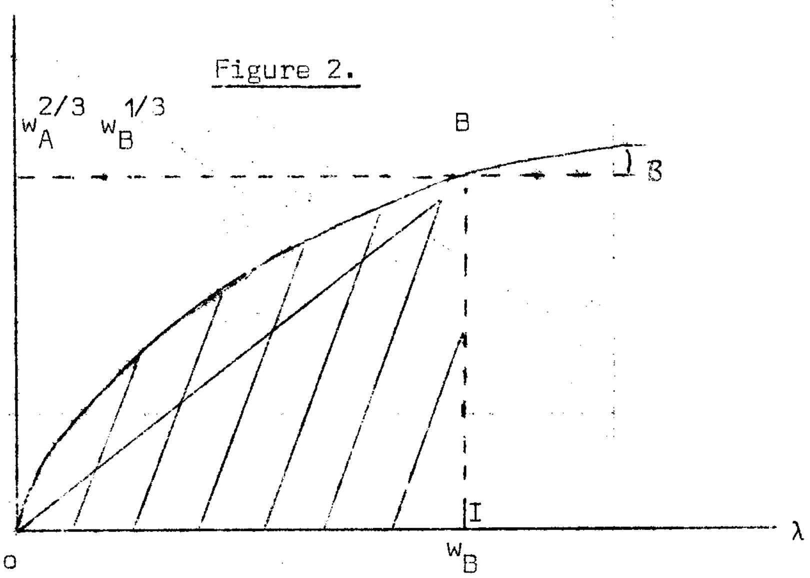


Figure 2.

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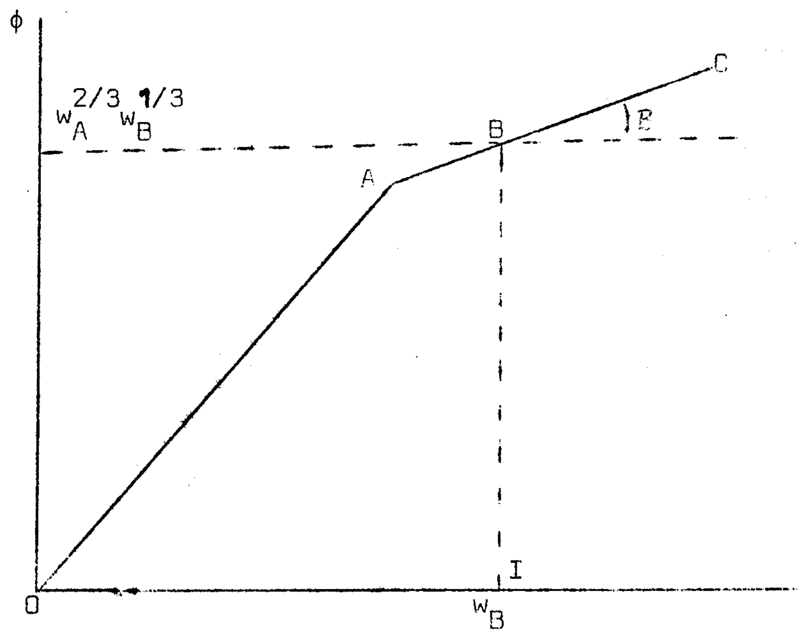
Figure 3.



This suggests modifying slightly the utility function along the following lines. Let us consider ϕ' , the graph of which is made of OA and of the half line ABC (Figure 3). The slope of ABC is β with $\mathrm{tg}\,\beta = (1/3)\ (w_A/w_B)^{2/3}$, so that the graphs of ϕ and ϕ' are tangent in

$$B : \begin{pmatrix} w_B \\ w_A^{2/3} \quad w_B^{1/3} \end{pmatrix} .$$

Let us consider now the (unique) function defined on $\mathbb{R}_+^2$ which is homogeneous of degree one and whose section by the plane $\mu = w_A$ coincides with ϕ' and let us denote it u'. u, the COBB-DOUGLAS utility function of parameter 2/3 and u' are such that in any point $x \in \mathbb{R}_+^2$, $(x = (\mu,\lambda))$ such that $\mu/\lambda = w_A/w_B$ the two $\mathbb{R}^3$ surfaces defined respectively by $z = u(x)$, $z' = u'(x)$ are <u>tangent</u>. (Equivalently, one can say that the graphs of u and u' are tangent along a half line of $\mathbb{R}_+^3$).

Let us thus consider the two goods economy of section III.A in which the common utility function of the agents would be u'. Let us term it the u'-economy as opposed to the economy in which the utility function is the COBB-DOUGLAS function of parameter 2/3 which will be referred to as the u-economy.

Let us look, in an EDGEWORTH box, at the indifference curves of consumers A et B in both u-economy and u'-economy. In both economies the indifference curves of consumers A and B have a common tangent along the diagonal of the EDGEWORTH box and <u>these common tangents coincide</u> in u and u' economies.

It turns out that the unique competitive equilibrium of the u-economy is also a competitive equilibrium of the u'-economy. Moreover, if there were more than one competitive allocation in the u'-economy[11], all would be equivalent in terms of utility for all agents. Consequently, the level of utility given to a member of the syndicate in a competitive equilibrium of the u'-economy is that associated with the bundle

$$\begin{pmatrix} \frac{2}{3} w_A \\ \frac{2}{3} w_B \end{pmatrix} .$$

On the other hand, the limit Shapley value of the u'-economy is given by formula II.B(3)

$$\lim_{n\to\infty} \frac{1}{n} Sh'_n(1) = \int_0^1 u'(w_A, tw_B)\, dt .$$

But obviously, one can choose the point A in order to make OAB arbitrarily close to OB, so that $\int_0^1 u'(w_A, tw_B)dt$ can be made arbitrarily close to the value of the surface of the triangle OIB times $1/w_B$, i.e.,

$$w_A^{2/3} w_B^{1/3} \frac{w_B}{2w_B} = \frac{1}{2} w_A^{2/3} w_B^{1/3} .$$

Hence, the bundle corresponding to the limit Shapley value can be made —through the choice of u'— arbitrarily close to

$$\begin{pmatrix} \frac{1}{2} w_A \\ \frac{1}{2} w_B \end{pmatrix} ,$$

a bundle which is less good for consumers of type A than the competitive bundle.

One can summarize :

Given the two goods - two consumers economy of section III.A, where the COBB-DOUGLAS utility function has been replaced by an "appropriate"[12] function u', the syndicate of all consumers A is not weakly A-stable in an n-replica of this economy, when n is large enough.

Let us finally notice that results similar to those presented in II-B, III-A, III-B are true using a measure space of agents instead of replicas.

The value is defined as in AUMANN [1974] using as value on pFL the one with uniform distribution in HART [1973].

C/ Some comments.

(a) The above example points out some specific properties of the limit Shapley value in an economy which is not "non-atomic"(13).

- A limit Shapley value does not coincide with a competitive equilibrium.

- A limit Shapley value is affected by "modification" of the preferences of the agents which do not affect competitive equilibria. Although the "modifications of preferences" considered in the above example were modifications of the network of indifference curves (outside a neighborhood of this part of the diagonal of the EDGEWORTH box which is not too close to zero) this latter fact strongly suggests that a limit Shapley value of an "atomic economy does depend on the particular cardinal utility functions chosen for representing the preferences of the agents. Thus the Shapley value of an atomic economy would be a "cardinal" concept when it is an "ordinal" concept in non atomic economy.

However, the cardinal representation of the preference preorderings chosen in the examples of paragraph 2 cannot be changed without making the corresponding economy leave out the class of "homogeneous markets". So our framework is inadequate for building an example supporting the preceding assertion.

(b) In the example of III.B the syndicate is not A-stable when the price system of the competitive equilibrium is favorable to owners of commodity 1. When this price system is less favorable to them (case $\alpha \leq 1/2$), the syndicate becomes again advantageous.

It is worth noting that the bundles of commodities given to syndicate members by the limit Shapley value are more concentrated when α varies than the corresponding competitive bundles. Moreover, the limit Shapley value gives the syndicate always more than half of the total resources of the economy, which is in some sense a "good" bundle, even if it is not always as good as the competitive equilibrium. This latter property

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is not linked to the COBB-DOUGLAS assumption, as can be seen by looking at formula II.B(3)

$$\lim \frac{1}{n} Sh_n(1) = \int_0^1 u(w_A, tw_B)\, dt - \int_0^1 u(tw_B)\, dt .$$

If $u(tw_B) = 0 \quad \forall\, t \geq 0$,

$$\lim \frac{1}{n} Sh_n(1) = \int_0^1 u(w_A, tw_B)\, dt$$

and since u is concave,

$$\frac{1}{n} Sh_n(1) \geq \frac{1}{2} u(w_A, w_B)$$

(supposing $u(w_A, 0) = 0$).

So the fact that the syndicate obtains for the limit Shapley value more than half of the total resources in the economies of paragraph II.B depends on the fact that it is a monopoly and that the good it owns has no substitute ($u(tw_B) = 0$).

This suggests that even if it does not imply A-stability, the Shapley value captures in some way the strength of monopolists owning a commodity which has no substitute. However attempts for exploring this idea, seemed to indicate that it was unlikely to be fruitful when economic agents do not have similar preferences[14].

IV - AN EXAMPLE OF A SYNDICATE WHICH IS NOT B-STABLE.

In order to exhibit a syndicate which is not B-stable, we will comme back to the economy of paragraph II.B[15]. We will suppose that in the n-replica of the initial two agents economy :

p agents of type A constitute a syndicate.

(n-p) agents of type A remain isolated.

n agents of type B are not syndicated.

There will be 2n-p+1 agents, having the same utility function u.

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Agent n° 1 is the syndicate. His vector of initial endowments is pw_A.

Agents n° 2 to n° n-p+1 are agents of type A having an initial endowment w_A.

Agents n° n-p+1 to 2n-p+1 are agents of type B having an initial endowment w_B.
When n will tend to infinity, we will require that the relative size of the syndicate p/n tends to a constant number.

The orderings of the 2n-p+1 players can be exhaustively described as follows :

player 1 is ranked 1st : there are (2n-p)! such orderings.
player 1 is ranked 2nd. Either
one player of type A is number 1 or
one player of type B is number 1
. . . .
player 1 is ranked q-th and ℓ players of type A and q-1-ℓ players of type B precede him.
There are $(2n+1-p-q)!(q-1)!C_\ell^{n-p}\ C_{q-1-\ell}^n$ orderings of this type.
. . . .

The Shapley value of player 1 in this replica economy can then be written down :

$$Sh_n(1) = \frac{1}{(2n-p+1)!} \sum_{q=1}^{q=2n-p+1} (q-1)!(2n+1-p-q)! \sum_{\ell=0}^{\ell = \min\begin{cases} n-p \\ q-1 \end{cases}} C_\ell^{n-p} C_{q-1-\ell}^n \Delta_{q\ell} \qquad (4)$$

where $\Delta_{q,\ell} = u((p+\ell)\ w_A\ ,\ (q-1-\ell)\ w_B) - u(\ell w_A, (q-1-\ell) w_B)$
(the argument is exactly similar to this in section II.B).

To compute the Shapley value associated with nonsyndicated player of type A, one has to consider again the set of orderings according to the rank of this anonymous player (that we will call player A).

Player A is q-th and ℓ players of type A and q-1-ℓ player B precede him.

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There are $(q-1)!(2n-p+1-q)!C_{\ell}^{n-p-1}C_{q-1-\ell}^{n}$ such permutations.

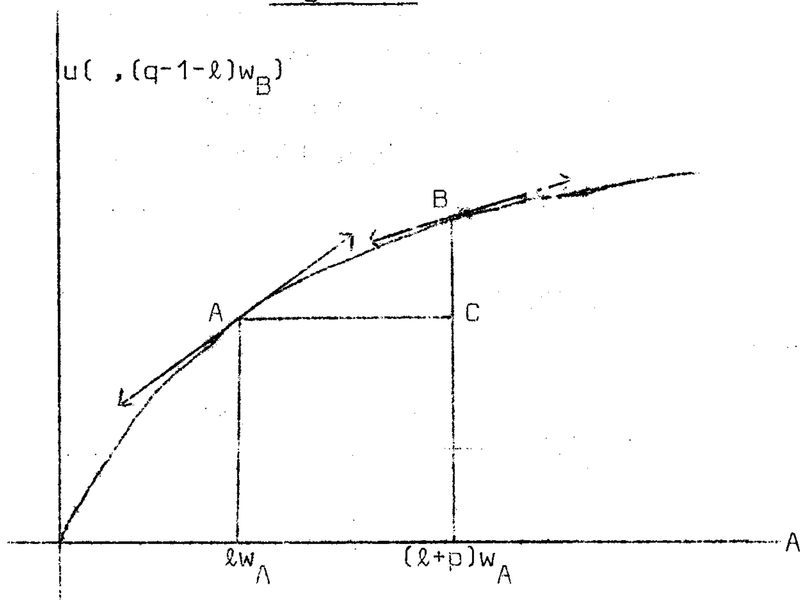
Player A is q-th and player 1 precedes him with $(\ell-1)$ players of type A and $(q-1-\ell)$ players of type B.
There are $(q-1)!(2n-p+1-q)!C_{\ell}^{n-p-1}C_{q-1-\ell}^{n}$ such orderings.

As $C_{\ell}^{n-p-1} = C_{\ell}^{n-p} \times \frac{n-p-\ell}{n-p}$ and $C_{\ell-1}^{n-p-1} = C_{\ell}^{n-p}\frac{\ell}{n-p}$, it becomes :

$$Sh_n(i) = \frac{1}{(2n-p+1)!}\sum_{q=1}^{q=2n-p+1}(q-1)!(2n+1-p-q)! \sum_{\ell=0}^{\ell=\min\begin{cases}n-p\\ q-1\end{cases}} C_{\ell}^{n-p}C_{q-1-\ell}^{n}\left(\frac{(n-p-\ell)\,\Delta'_{q,\ell}+\ell\Delta''_{q,\ell}}{n-p}\right) \quad (5)$$

$i = 2,\ldots,n-p+1$

Figure 4.



where

$$\Delta'_{q,\ell} = u((\ell+1)w_A\ ,\ (q-1-\ell)w_B) - u(\ell w_A,(q-1-\ell)w_B)$$

$$\Delta''_{q,\ell} = u((\ell+p)w_A\ ,\ (q-1-\ell)w_B) - u((\ell+p-1)w_A,\ (q-1-\ell)w_B)\ .$$

One notes that the formula giving $Sh_n(i)$ can be obtained by replacing in the formula giving $Sh_n(1)$ $\Delta_{q,\ell}$ by

$$\left(1-\frac{\ell}{n-p}\right)\Delta'_{q,\ell} + \left(\frac{\ell}{n-p}\right)\Delta''_{q,\ell}\ ,$$

so that the comparison of the Shapley values of the syndicated and non syndicated player rest upon the comparison of $\frac{\Delta_{q,\ell}}{p}$ and $(1 - \frac{\ell}{n-p})\,\Delta'_{q,\ell}$ + $\frac{\ell}{n-p}\,\Delta''_{q,\ell}$. In order to have an intuitive understanding of the situation, let us look at Figure 4, in which the graph of the function $u(\lambda,(q-1-\ell)w_B)$ is drawn and in which w_A is put equal to 1. Then $\Delta'_{q,\ell}$, $\Delta''_{q,\ell}$ are respectively (nearly) the slopes of the tangents in A and B when $\frac{1}{p}\,\Delta_{q,\ell}$ is the slope of AB.

So the comparison involved by formulas (4) and (5) rests, roughly speaking, upon the "curvature" of the graphs such as that of Figure 4 and of the relative weights of $\Delta'_{q,\ell}$ and $\Delta''_{q,\ell}$, —the weight of $\Delta'_{q,\ell}$ decreasing as ℓ increases—.

One can guess that if the graph of u is as depicted in Figure 5 (with ε small), the mean of the contributions of an unorganized agent will tend to become greater than the contribution of a member of the syndicate. (According to formulas (4) and (5), for ℓ large both agents will contribute equally, for ℓ small, the small unorganized agent will contribute more). Let us make this idea more precise.

Let us suppose that u is such that its section by the plane $\mu = w_B$ is as indicated in Figure 6. For $0 \le v \le w_A/100$, $u(\lambda,w_B) = 2\lambda$. For $v \ge w_A/100$, $u(\lambda,w_B) = w_A/100 + \lambda$.

From the homogeneity of u, its section by the plane $\mu = (q-1-\ell)w_B$ will be as indicated in figure 7.

Let us consider q as given and let us compare $\Delta_{q,\ell}$ and $E_{q,\ell} = (1-\ell/(n-p))\,\Delta'_{q,\ell} + (\ell/(n-p))\,\Delta''_{q,\ell}$ when ℓ varies.

Case(i) : $\ell \ge (q-1-\ell)/100$.

Then $\Delta_{q,\ell}/p = E_{q,\ell}$.

Case (ii) : $\ell < (q-1-\ell)/100$.

Then $\ell < (q-1)/100$ and

$$\Delta_{q,\ell}/p \le w_A[1 + \frac{q-1}{p} \cdot \frac{1}{100}]$$

$$E_{q,\ell} = w_A\,[1 + (1 - \frac{\ell}{n-p})] \quad .$$

Hence $E_{q,\ell} > \Delta_{q,\ell}/p$ will follow from

$$1 - \frac{\ell}{n-p} > \frac{q-1}{p} \cdot \frac{1}{100}$$

Figure 5

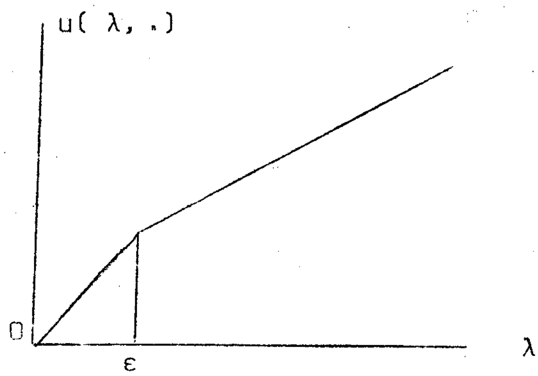


Figure 6

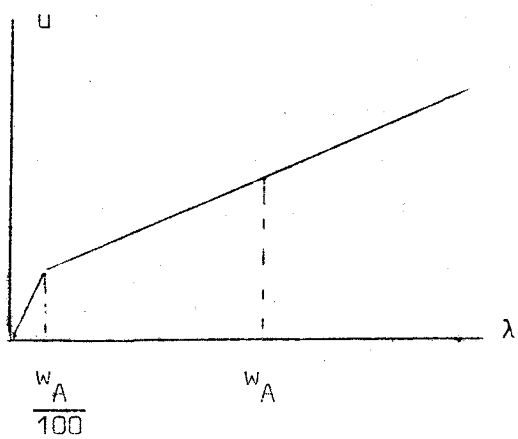
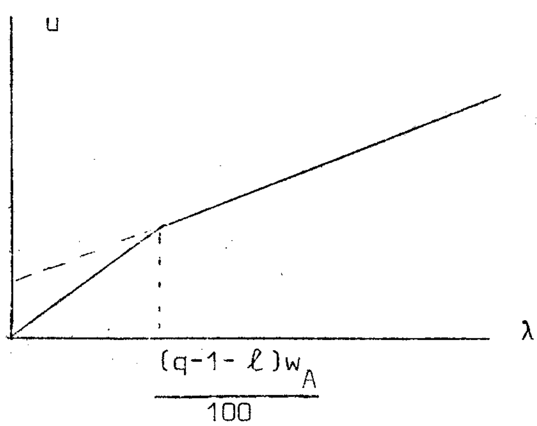


Figure 7



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which follows from

$$\frac{2 - (\frac{p}{n})}{(\frac{p}{n})(1 - (\frac{p}{n})} < 100 \quad ;$$

the last inequality holds for p/n not too close to 0 or to 1.

So with the special utility function considered and with for example $p/n = 2/3$, $E_{q,\ell}$ is always greater and sometimes strictly greater than $\Delta_{q,\ell}$: the Shapley value of the small agent is greater than the Shapley value of a member of the syndicate. One can state :

Let us consider an n-replica of the above economy in which p agents of type A are syndicated. Then for every p, n such that p/n not too close to zero or one , the syndicate is not B-stable.

CONCLUSION.

From this analysis, two main conclusions can be emphasized. The first one is a negative answer : syndicates cannot generally be expected to be either A-stable or B-stable with respect to the solution concept of Shapley value. The examples we have built have no pathological features. Moreover, they are borrowed from a class of economies for which syndicates are stable with respect to the solution concept of core. The second one is that the Shapley value of non-atomic economies does not remain independent of the cardinal representations of agents'utilities. This remark may lead to a less pessimistic view on the problem of stability of syndicates. Certainly, no "general" stability can be expected, but there is some hope for finding out simple cardinal characteristics which would be crucial for stability. This is a field open to future research.

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Appendix :

Some Statements in a Measure Space of Economic Agents.

The above analysis of A-stability suggests few "positive" statements. However, the arguments of paragraph as well as the remark can be made more systematic.

For that, let us consider a measure space of economic agents $(T, \mathcal{C}, \mu)$. T, the set of agents, is supposed to be the interval $[0,N]$. The agents fall into N types (the types will be denoted by $i = 1,\ldots,N$), all agents of each type having the same initial endowments and the same preferences. Agents of type i belong to the interval $[i-1, i[\; = T_i$. Agents of type N belong to the interval $[n-1, n] = T_n$.

$\mathcal{C}$ is the class of Lebesgue measurable subsets and μ the Lebesgue measure. The commodity space is $\mathbb{R}^{\ell}_{+}$. This classical measure space of economic agents with types (cf. Gabzewicz-Drèze (1971)) will be now specified as follows.

a. All agents $t \in T$ have the same cardinal utility function u defined on $\mathbb{R}^{\ell}_{+}$. u has the following properties (H1) :

u is measurable on the σ-field of Borel subsets of $\mathbb{R}^{\ell}_{+}$.

$u(x) = o\,\|x\|$ when $\|x\| \to +\infty$, i.e., $|u(x)|/\|x\| \to 0$, $\|x\| \to +\infty$.

u is concave and increasing and $u(o) = 0$.

u has continuous partial derivatives $\partial u/\partial x_j$, $\forall\, x \in \mathbb{R}^{\ell}_{+} \mid x_j > 0$.

b. All agents of type i have the same initial endowment $w_i \neq 0$ $i = 1,\ldots,N$.

c. Agents of type 1 form a syndicate : the set of potential coalitions is a subset of $\mathcal{C}$, $\mathcal{C}_1$ defined as follows :

$\mathcal{C}_1 = \{S \in \mathcal{C} \mid \text{either } S \cap T_1 = \phi, \text{ or } S \cap T_1 = T_1\}$. Generally the game defined above is a game without transferable utility. We will define a Shapley value in a way similar to that of paragraph 2. <u>A priori</u>, the weights λ_i of paragraph 2 should be replaced by a measure $\lambda(t)$. However, as we require that the

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Shapley value without transferable utility we are defining be symmetric, we will restrict our attention to the degenerated measures such that $\lambda(t) = \lambda(t')$, $\forall\, t,t' \in T_i$, $\forall\, i$. Precisely, given $\lambda \in S^N$ $\{S^N = \lambda = \lambda_1,\ldots,\lambda_N) \mid \sum_1^N \lambda_i = 1, \lambda_i \geq 0\}$ one will define the game G_λ whose characteristic function $v(\lambda,.)$ is defined for every $S \in \mathcal{C}_1$ by

$$v(\lambda,S) = \text{Max} \sum_{i=1}^{N} \int_{T_i \cap S} \lambda_i\, u(x(t))d\mu(t)$$

$$\sum_{i=1}^{N} \int_{T_i \cap S} x_i(t)d\mu(t) \leq \sum_{i=1}^{N} \Big(\int_{T_i \cap S} d\mu(t)\Big)w_i$$

$$x_i(t) \in \mathbb{R}_+^{\ell}\ , \ \forall\, t \in T.$$

Let us consider $y = (y_1,\ldots,y_N)$ and

$$g(\lambda,y,z) = \text{Max} \sum_{i=1}^{N} \lambda_i\, y_i u(x_i)$$

$$\sum_{i=1}^{N} y_i\, x_i \leq z \quad , \quad x_i \in \mathbb{R}_+^{\ell} \quad , \quad \forall\, i = 1,\ldots,N.$$

According to Lemma 39.8 of Aumann-Shapley (1974), and given the assumptions made on u, $v(\lambda,S) = g(\lambda,y,z)$ with $y_i = \int_{T_i \cap S} d\mu(t)$ and $\sum_{i=1}^{N} y_i\, w_i = z$.

But as in Lemma 39.16 of Aumman-Shapley (1974), $v(\lambda,S)$ can be written $g(\lambda,\eta(S),\ \xi(S))$ where $\eta(S)$ is an N vector of measures and $\xi(S)$ an ℓ-vector of measures. Each component of these vectors is a measure which is itself the sum of measures on the non-atomic part of T and of a measure which has a finite carrier in the sense of Hart (1973). Furthermore, from Proposition 39.13 of Aumann-Shapley (1974) it comes out that g has continuous partial derivatives for $y > 0$, $z > 0$. Taking into account Proposition 10.17 of Aumann-Shapley (1974), $v(\lambda,.)$ is proved to belong to the class of set functions for which Hart proves the existence of a value. Actually in Hart's article (1973) an infinite number of values is proved to exist. Moreover, among them only one is the limit of the values of the sequence of finite games that

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one can associate with the infinite game. Only this value will be considered here.

We are ready to prove Proposition I, which extends the argument of Remark III.C(b).

Proposition 1 : Consider the economy defined above in which all agents have identical preferences satisfying H1. Let us suppose that the syndicate formed by agents of type 1 be a monopoly and that the commodity(ies) it owns has (have) no substitute in the sense that $u(w_2 + \dots w_N) = u(0) = 0$. Then, the Shapley value bundle of the members of the monopoly is envied by all other agents in the economy.

Proof : Let λ^* be a set of weights associated with a Shapley value of the game without transferable utility. The Shapley value of the game with transferable utility G_{λ^*} can be computed according to Hart's procedure (with a uniform probability distribution). It comes with the notations of paragraph 1 :

$$\lambda_1^* \; u_1^* = \int_0^1 [g(\lambda^*, y(\alpha), z(\alpha)) - g(\lambda^*, y'(\alpha), z'(\alpha))]dt$$

where

$$y(\alpha) = (1, \alpha, \alpha, \dots, \alpha) \quad , \quad z(\alpha) = w_1 + \alpha \sum_{i-2}^{N} w_i$$

$$y'(\alpha) = (0, \alpha, \alpha, \dots, \alpha) \; , \; z'(\alpha) = \alpha \sum_{i}^{N} w_i \; .$$

But as the commodities owned by 1 have no substitutes

$$\lambda_1^* \; u_1^* = \int_0^1 g(\lambda^*, y(\alpha), z(\alpha))dt.$$

But g is a concave function of α (it can be proved either directly or as resulting from Lemma 39.9 in Aumann-Shapley (1974). Hence $\int_0^1 g(\lambda^*, y(\alpha), z(\alpha)) \, d\alpha$ $\geq \frac{1}{2}[g(\lambda^*, y(1), z(1)) - g(\lambda^*, y(0), z(0))]$. It follows that $\lambda_1^* \; u_1^* \geq g(\lambda^*, y(1), z(1))/2 = (1/2) \sum_{i=1}^{N} \lambda_i^* \; u_i^*$. Hence

$$\lambda_1^* \; u_1^* \geq \sum_{i=2}^{N} \lambda_i^* \; u_i^* \; .$$

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Let us remember that $\sum_{i=1}^{N} \lambda_i^* u_i^*$ is the maximum of the linear form $\lambda_i^* u_i$ on a set of feasible utility which is symmetric (because all agents have identical preferences). We will prove $u_1^* \geq u_i^*$ $\forall$ i = 2,...,N. First let us prove that $\lambda_1^* \geq \lambda_i^*$ $\forall$ i = 2,...,N. Suppose that there exist i s,t, $\lambda_1^* < \lambda_i^*$; thus $u_1^* < u_i^*$ (from the symmetry recalled above), and $\lambda_1^* u_1^* < \lambda_i^* u_i^*$ which contradicts the formula. But $\lambda_1^* \geq \lambda_i^*$ $\forall$ i = 2,...,N, implies $u_1^* \geq u_i^*$ $\forall$ i = 2,...,N (with the same symmetry argument). Q.E.D.

Let us now turn to the case where u, the utility function, is Cobb-Douglas. Precisely, let us replace H1 by H2 :

$$\text{If } x = (x_1,\dots,x_e),\ u(x) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_e^{\alpha_e},\ \alpha_i > 0\ \sum \alpha_i = 1.$$

Although u does not satisfy one of the conditions required by H1, one can prove that every game G associated with this economy has a Shapley value : for this, it suffices to remark that the reasonings of paragraph 2 concerning the shape of the set of feasible utilities in a homogeneous market can be transposed here and imply $g(\lambda,y,z) = (\text{Max}_i\ \lambda_i)u(z)$. The existence of a Shapley value for G follows from reasonings similar to those used earlier when u satisfied H1.

We can now state Proposition 2 which extends the analysis of section IIIA.

<u>Proposition 2</u> : Let us consider the above economy in which the utility function of all agents satisfies H2.

If the syndicate formed by agents of type 1 is a monopoly owning the total quantity of one commodity (and a zero quantity of all other commodities), then it is (strongly) A-stable with respect to the solution concept of Shapley value.

Proof : The details are left to the reader. The outline of the proof is as follows :

a) As in paragraph 2, it can be proved that the only possible λ^* defining a Shapley value of the game without transferable utility is $\lambda^* = (1/N, 1/N, \dots, 1/N)$

b) Applying Hart's formula, and taking into account the above remark on $g(\lambda, y, z)$, it comes out

$$u_1^* = \int_0^1 u(w_1 + t(w_2 + \dots w_N) dt \qquad (u(t(w_2 + \dots w_N) = 0).$$

If the monopoly commodity is commodity number one and with straightforward notations it becomes :

$$u_1^* = \int_0^1 (w_{11})^{\alpha_1} \dots (\sum_{i=2}^N w_i)_j^{\alpha_j} \dots (\sum_{i=2}^N w_i)_e^{\alpha_e} \, t^{\alpha_2 \dots \alpha_e} \, dt$$

$$u_1^* = u(\sum_{i=1}^N w_i) \int_0^1 t^{1-\alpha_1} dt = \frac{1}{2-\alpha_1} \, u(\sum_{i=1}^N w_i).$$

c) The competitive equilibrium gives to agent 1 the bundle $\bar{\bar{x}}_1 = \alpha_1 (\sum_{i=1}^N w_i)$ (16). The conclusion follows as in paragraph III.A.

FOOTNOTES.

(1) The economic game considered in the following will meet these requirements.

(2) For example, the notion of weak A-stability arbitrarily refers to the point of view of potential syndicate members questioning about the creation of a syndicate and not to the point of view of actual syndicate members considering to leave out the syndicate.

(3) $\forall\ i \in I$ should be replaced by "for almost every $i \in \tilde{I}$," and $\exists\ j \in J/I$ by "$\exists\ A_i \subset J/\bar{I}$ s.t. $\mu(A_i) > 0$" (straightforward notations).

(4) Our concept of weak B-stability would then be close to the concept of weak marginal stability defined by GABZEWICZ-DREZE for solution concept of core [1971]. However, in GAEZEWICZ-DREZE weak marginal stability as well as strong total stability are properties of allocations, when A-stability or B-stability are here properties of a syndicate.

(5) Cases where XS = X can be found in SHITOVITZ [1973], DREZE-GABZEWICZ-SCHMEIDLER-VIND [1972].

(6) At contrary the measure space approach is more appropriate for proving géneral statements such that those few "positive" statements indicated in the following.

(7) For more details on the λ-transfer procedure, see SHAPLEY [1969].

(8) The Shapley value in an exchange economy can be seen as resulting from a confrontation of "supply" and "demand" of utility levels :
- the "supply of utility levels, when the weights of the individuals are $\lambda = (\lambda_1,\ldots,\lambda_n)$, is a vector $\bar{u}(\lambda)$ such that $\sum_i \lambda_i \bar{u}_i(\lambda)$ $= \text{Max} \sum \lambda_i u_i(x_i)$, $\sum_{i \in N} x_i \leq \sum_{i \in N} w_i$.

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- the "demand" of utility levels from the agents is the vector $\bar{\bar{u}}(\lambda)$ s.t. $\bar{\bar{u}}_i(\lambda) = Sh_i(\lambda)/\lambda_i$ where $Sh_i(\lambda)$ is given by formula (1).

This remark immediately suggests a way of proving an existence theorem for the Shapley value in an exchange economy. Such a theorem is given in CHAMPSAUR [1971].

(9) Given our assumptions on u this integral does exist.

(10) This results from the convergence theorem proved by CHAMPSAUR [197_5].

(11) i.e., if u' were not "strictly" quasi-concave (with a straightforward meaning).

(12) i.e., s.t. the slope of OA be close to the slope of OB.

(13) Here, we use the word atomic in the context of a replica-economy, with a straightforward meaning.

(14)

(15) This can be seen, for example, by considering the competitive equilibrium the Shapley value of the non-atomic game. According to AUMANN-SHAPLEY [1974], Theorem B, it becomes $\bar{\bar{u}}_1 = \int_0^1 \frac{\partial u}{\partial x_1} (t(\sum_1^N w_i))w_{11}dt$ $= \int_0^1 \frac{\partial u}{\partial x_1} (\sum_1^N w_i)w_{11}dt$. As $\frac{\partial u}{\partial x_1} (\sum_1^N w_i) \times w_{11} = \alpha_1 \; u(\sum_1^N w_i)$, it becomes

$\bar{\bar{u}}_1 = \alpha_1 \; u(\sum_1^N w_i)$. Q.E.D.

(16) As mentioned in the introduction, we are considering here a replica economy rather than a measure space of agents, despite the greater complexity of the analysis in this framework. The reason is that HART's formula [1973] does not apply directly to the agents of the non-atomic part of the economy. Extending HART's analysis for our problem would involve a specific work which is outside our scope.

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